Uniqueness and complexity in generalised colouring

by

Alastair Farrugia

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The study and recognition of graph families (or graph properties) is an essential part of combinatorics. Graph colouring is another fundamental concept of graph theory that can be looked at, in large part, as the recognition of a family of graphs that are colourable according to certain rules.

In this thesis, we study additive induced-hereditary families, and some generalisations, from a colouring perspective. Our main results are:

- Additive induced-hereditary families are uniquely factorisable into irreducible families.

- If $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary graph families, then $(\mathcal{P}, \mathcal{Q})$-COLOURING is NP-hard, with the exception of GRAPH 2-COLOURING. Moreover, with the same exception, $(\mathcal{P}, \mathcal{Q})$-COLOURING is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both in NP. This proves a 1997 conjecture of Kratochvíl and Schiermeyer.

We also provide generalisations to somewhat larger families. Other results that we prove include:

- a characterisation of the minimal forbidden subgraphs of a hereditary property in terms of its minimal forbidden induced-subgraphs, and *vice versa*;

- extensions of Mihók's construction of uniquely colourable graphs, and Scheinerman's characterisations of compositivity, to disjoint compositive properties;

- an induced-hereditary property has at least two factorisations into arbitrary irreducible properties, with an explicitly described set of exceptions;

- if $\mathcal{G}$ is a generating set for $\mathcal{A} \circ \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are indiscompositive, then we can extract generating sets for $\mathcal{A}$ and $\mathcal{B}$ using a "greedy algorithm".

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My contacts with Vadim Lozin in the last few months were particularly serendipitous, as they spurred me to look at the complexity issue from a different angle, and led to the proof of Theorem 6.1.4.

I would like to thank Therese Biedl, Jim Geelen, Bertrand Guenin and Jan Kratochvíl for forming my thesis committee, and for their careful reading and helpful comments. Therese's course on graph theoretic algorithms was both enjoyable and instructive, as that is where I learnt about intersection graphs and compositivity. Jim's simpler proof of Theorem 4.3.4 is included in the thesis.

I was lucky to meet several people during my three years in Waterloo, particularly in the Department of Combinatorics, and at WCRI (Waterloo Co-operative Residence Inc.) and WPIRG (Waterloo Public Interest Research Group). I would like to mention in particular Jacqui Williamson, Evie and Matthew Hill, Zhade Thompson, Kim Honeyford, room-mates Kathryn Laird and Ryan Stoughton, office-mates Jonathan Dumas, Jin Qian and Maya

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Dedication

I dedicate this thesis to

my father, Joe Farrugia Cassano,

who taught me the joy of arithmetic and mathematics,
and other things that are far more important.

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$G \subseteq H$	G is a subgraph of H	2
$G \subset\!\!\cdot\; H$	G is a spanning subgraph of H	46
$G \leq H$	G is an induced-subgraph of H	2
$G \preceq H$	G precedes H in the partial order $\preceq$	12
$G \cup H$	the vertex-disjoint union of G and H	2
$G + H$	the join of G and H	32
$\mathcal{A} + \mathcal{B}$	$\{G + H \mid G \in \mathcal{A}, H \in \mathcal{B}\}$	45
$G * H$	the $*$-join of G and H, $\{J \mid (G \cup H) \subseteq J \subseteq (G + H)\}$	52
$\mathcal{P} \circ \mathcal{Q}$	the product of $\mathcal{P}$ and $\mathcal{Q}$, $\bigcup\{G * H \mid G \in \mathcal{P}, H \in \mathcal{Q}\}$	3
$s \circledast G$	$G * \cdots * G$, the graphs spanned by s disjoint copies of G	53
$\mathcal{G}_{\preceq}$	the smallest $\preceq$-hereditary property containing $\mathcal{G}$	13
$\mathcal{G}_{\subseteq}$	the smallest hereditary property containing $\mathcal{G}$	36
$\mathcal{G}_{\leq}$	the smallest induced-hereditary property containing $\mathcal{G}$	52
$\mathcal{G}^{a}$	the smallest additive property containing G	13
$\mathcal{G}^{a}_{\preceq}$	the smallest additive $\preceq$-hereditary property containing $\mathcal{G}$	13
$\mathcal{F}_{\subseteq}(\mathcal{P})$	the set of minimal forbidden subgraphs for $\mathcal{P}$	12
$\mathcal{F}_{\leq}(\mathcal{P})$	the set of minimal forbidden induced-subgraphs for $\mathcal{P}$	12
$\mathrm{Forb}_{\subseteq}(S)$	$\{G \mid \forall H \in S,\ H \not\subseteq G\}$	31
$\mathrm{Forb}_{\leq}(S)$	$\{G \mid \forall H \in S,\ H \not\leq G\}$	31
$(\mathcal{G}_1 + \cdots + \mathcal{G}_m)^{\downarrow}$	$\{G' \in \mathcal{M}^*(\mathcal{P}) \mid dc(G') = dc(\mathcal{P}),$ and $\exists\, G \in \mathcal{G}_1^{\downarrow} + \cdots + \mathcal{G}_m^{\downarrow},\ G \subset\!\!\cdot\; G'\}$	46
$(\mathcal{G}_1 * \cdots * \mathcal{G}_m)^{\downarrow}$	$\{G' \in \mathbf{S}(\mathcal{P}) \mid dec_{\mathcal{P}}(G') = dec(\mathcal{P}),$ and $\forall\, i,$ $1 \leq i \leq m,\ \exists\, G_i \in \mathcal{G}_i^{\downarrow},\ G' \in G_1 * \cdots * G_m\}$	71

Symbol	Definition	Page
$dc(G)$	decomposability of G, number of components of $\overline{G}$	37
$dc(\mathcal{G})$	$\min\{dc(G) \mid G \in \mathcal{G}\}$	38
$dc(\mathcal{P})$	$dc(\mathcal{M}^*(\mathcal{P}))$	38
$c(\mathcal{P})$	the completeness of $\mathcal{P}$, $\max\{n \mid K_n \in \mathcal{P}\}$	36
$\mathcal{M}(\mathcal{P})$	the set of $\mathcal{P}$-maximal graphs	37
$\mathcal{M}^*(\mathcal{P})$	the set of $\mathcal{P}$-strict $\mathcal{P}$-maximal graphs	37
$\mathcal{G}^{\downarrow}$	$\{G \in \mathcal{G} \mid dc(G) = dc(\mathcal{P})\}$	39
$\mathcal{G}[H]$	$\{G \in \mathcal{G} \mid H \subseteq G\}$	23
$Ip(G)$	the ind-parts of G	40
$I_{\mathcal{G}}$	$\cup(Ip(G) \mid G \in \mathcal{G})$	40
$d\|G$	the decomposition induced by d on G	57
$dec_{\mathcal{P}}(G)$	the $\mathcal{P}$-decomposability of G	55
$dec_{\mathcal{P}}(\mathcal{G})$	$\min\{dec_{\mathcal{P}}(G) \mid G \in \mathcal{G}\}$	55
$dec(\mathcal{P})$	$dec_{\mathcal{P}}(\mathbf{S}(\mathcal{P}))$	55
$\mathcal{G}^{\downarrow}$	$\{G \in \mathcal{G} \mid dec_{\mathcal{P}}(G) = dec(\mathcal{P})\}$	56
$\mathcal{G}[H]$	$\{G \in \mathcal{G} \mid H \leq G\}$	21
$\mathbf{S}(\mathcal{P})$	the set of $\mathcal{P}$-strict graphs	55
$\mathbf{S}^{\downarrow}(\mathcal{P})$	the set of $\mathcal{P}$-strict graphs G with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$	68
$\mathbf{S}^{\Downarrow}(\mathcal{P})$	the set of $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graphs G with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$	68
$\mathbb{U}$	the class of all properties	4
$\mathbb{L}$	hereditary properties	3
$\mathbb{L}_{\leq}$	induced-hereditary properties	3
$\mathbb{H}$	hom-properties	6
$\mathbb{L}^a$	additive hereditary properties	3
$\mathbb{L}^a_{\leq}$	additive induced-hereditary properties	3
$\mathbb{L}^c$	hereditary compositive properties	18
$\mathbb{L}^{dc}$	hereditary disjoint compositive properties	18
$\mathbb{L}^c_{\leq}$	induced-hereditary compositive properties	18
$\mathbb{L}^{dc}_{\leq}$	induced-hereditary disjoint compositive properties	18
$\overline{\mathcal{P}}$	$\{\overline{G} \mid G \in \mathcal{P}\}$	80
$\overline{\mathbb{P}}$	$\{\overline{\mathcal{P}} \mid \mathcal{P} \in \mathbb{P}\}$	80
$\mathbb{P}_1 \circ \mathbb{P}_2$	$\mathbb{P}_1 \cup \mathbb{P}_2 \cup \{\mathcal{P}_1 \circ \mathcal{P}_2 \| \mathcal{P}_i \in \mathbb{P}_i\}$	100

Chapter 1

Introduction

The study and recognition of graph families is an important part of combinatorics. Graph colouring is another fundamental concept of graph theory that can be looked at, in large part, as the recognition of families of graphs that are colourable according to certain rules. It is natural to consider additive induced-hereditary families in a colouring context, and this class of families has in fact attracted a considerable amount of attention (cf. [11, 27]).

Our main contribution is to show that reducible additive induced-hereditary families are uniquely factorisable into irreducible families, and that, with one exception, it is NP-hard to recognise them.

For example, let $\mathcal{L}$ and $\mathcal{F}$ be the sets of line-graphs and forests, respectively. Let $\mathcal{L} \circ \mathcal{F}$ be the set of graphs G such that $V(G)$ partitions into $V_{\mathcal{L}}$ and $V_{\mathcal{F}}$, with $G[V_{\mathcal{L}}] \in \mathcal{L}, G[V_{\mathcal{F}}] \in \mathcal{F}$. Then:

(i) $\mathcal{L} \circ \mathcal{F}$ is the only way to express this set as a product of additive induced-hereditary properties, i.e. if $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are additive induced-hereditary properties such that

$$[G \in \mathcal{L} \circ \mathcal{F}] \Leftrightarrow [V(G) = \cup_{i=1}^{m} V_i, G[V_i] \in \mathcal{A}_i],$$

then $\{\mathcal{A}_1, \ldots, \mathcal{A}_m\} = \{\mathcal{L}, \mathcal{F}\}$; and

(ii) it is NP-complete to determine if $G \in \mathcal{L} \circ \mathcal{F}$.

The results presented in this thesis appear in [4, 31, 32, 33]. The thesis itself is available at http://etheses.uwaterloo.ca. In the rest of this chapter we give some basic definitions and historical background, and describe our results in more detail.

1.1 Additive induced-hereditary properties

We consider only finite, simple graphs in this thesis — undirected, unlabeled, loopless graphs with at least one vertex but no multiple edges — apart from a few explicitly noted exceptions in Chapter 2. We refer the reader to [60] for graph theoretic concepts that are not defined here. A *graph property* $\mathcal{P}$ is a non-empty set, or family of graphs. The statements G *has property* $\mathcal{P}$ and G *is in* $\mathcal{P}$ are synonymous. For example, a graph is planar iff it is in the set of planar graphs. Unless explicitly stated otherwise, the properties we consider contain at least one, but not all non-null graphs.

We write $G \leq H$ when G is an induced-subgraph of H, that is, there exists $W \subseteq V(H)$ such that $G = H - W$. We write $G \subseteq H$ when G is a subgraph of H, that is, there exist $W \subseteq V(H)$ and $F \subseteq E(H)$ such that $G = (H - W) - F$. For graphs G and H, their *disjoint union* $G \cup H$ is defined by $V(G \cup H) := V(G) \cup V(H)$ (where $V(G) \cap V(H) = \emptyset$), and $E(G \cup H) := E(G) \cup E(H)$.

A property $\mathcal{P}$ is *hereditary* or *induced-hereditary* if, for any graph $G \in \mathcal{P}$, all its subgraphs or induced-subgraphs, respectively, are also in $\mathcal{P}$. A property is *additive* if for any two graphs $G, H \in \mathcal{P}$, where possibly $G \cong H$, their disjoint union is also in $\mathcal{P}$. Thus, for an additive induced-hereditary property $\mathcal{P}$, G is in $\mathcal{P}$ iff every component of G is in $\mathcal{P}$.

Note that some authors use 'hereditary' or 'monotone' to mean 'induced-hereditary'; and 'class' to mean 'property', whereas we will talk about, say, the 'class of hereditary properties'. We also note that every induced-hereditary property $\mathcal{P}$ is 'scarce' — if H is a graph not in $\mathcal{P}$, then almost all graphs contain H, but no graph in $\mathcal{P}$ does.

Let K_r be the complete graph on r vertices, $\overline{G}$ the complement of G, $q(G)$, $\Delta(G)$ and $\tau(G)$ the maximum component size, maximum degree and maximum path length of G, respectively, and let P_k be the path on k vertices. With this notation, we can give a few examples of induced-hereditary properties:

$$
\begin{array}{lcllcl}
\mathcal{O}_k & := & \{G \mid q(G) \leq k\} & \mathcal{O} & := & \{\overline{K}_r \mid r \geq 0\} \\
\mathcal{S}_k & := & \{G \mid \Delta(G) \leq k\} & \mathcal{K} & := & \{K_r \mid r \geq 0\} \\
\mathcal{W}_k & := & \{G \mid \tau(G) \leq k\} & \mathcal{L} & := & \{\text{line-graphs}\} \\
\mathcal{I}_k & := & \{G \mid K_k \not\leq G\} & \mathcal{B} & := & \{\text{perfect graphs}\} \\
\mathcal{T}_k & := & \{G \mid G \leq P_k\} & & &
\end{array}
$$

$$\begin{array}{lcl} H_{\subseteq} & := & \{\text{finite subgraphs of a fixed graph } H\} \\ H_{\leq} & := & \{\text{finite induced-subgraphs of a fixed graph } H\} \\ \to H & := & \{G \mid G \text{ is homomorphic to a fixed graph } H\}. \end{array}$$

These properties are all additive, except for $\mathcal{K}$, $\mathcal{T}_k$, $H_{\subseteq}$ and $H_{\leq}$. They are also hereditary, except for $\mathcal{K}$, $\mathcal{L}$, $\mathcal{B}$, $\mathcal{T}_k$ and $H_{\leq}$. The variety present even in this small sample shows the generality of the concepts we are dealing with; the advantages will become especially evident when we tackle complexity issues in Section 1.4 and Chapter 6.

1.2 Generalised colouring

A generalised colouring of a graph is an assignment of n colours to its vertices so that the i^{th} colour class (the set of vertices coloured i) induces a subgraph of G with some specified property $\mathcal{P}_i$ (cf. Figure 1.1, with $n = 3$). More formally, a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-*colouring* of G is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that, for all i, the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$ (unless $V_i = \emptyset$, which is also allowed). When $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{O}$ (where $\mathcal{O}$ is the set of finite edgeless graphs), we get traditional graph n-colouring.

The set of all $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable graphs is itself a property, denoted $\mathcal{P} = \mathcal{P}_1 \circ \ldots \circ \mathcal{P}_n$, or just $\mathcal{R}^n$ if $\mathcal{P}_1 = \ldots = \mathcal{P}_n = \mathcal{R}$. The $\mathcal{P}_i$'s are *factors* or *divisors* of $\mathcal{P}$, while $\mathcal{P}$ is the *product* of the $\mathcal{P}_i$'s.

It is easy to see that the product of additive (or hereditary or induced-hereditary) properties is also additive (or hereditary or induced-hereditary) — if the $\mathcal{P}_i$'s are all additive induced-hereditary, then $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourings of each component of G together give us a colouring of G, while the restriction of any $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colouring of G to a subset $U \subseteq V(G)$ is a colouring of the induced-subgraph $G[U]$. This means that $\mathbb{L}$ and $\mathbb{L}^a$, the classes of hereditary and additive hereditary properties, are closed under multiplication, as are their induced-hereditary analogues, $\mathbb{L}_{\leq}$ and $\mathbb{L}^a_{\leq}$. These classes are lattices [13], but we mention this only to explain the standard use of the letter $\mathbb{L}$.

Clearly we have $\mathbb{L}^a \subset \mathbb{L} \subset \mathbb{L}_{\leq}$ and $\mathbb{L}^a \subset \mathbb{L}^a_{\leq} \subset \mathbb{L}_{\leq}$. To see that the four containments are strict, consider properties $H_{\subseteq}$, $H_{\leq}$, $\mathcal{B}$ and $H_{\leq}$, respectively, where H is some fixed finite graph with at least one edge; $H_{\subseteq} \in \mathbb{L} \setminus \mathbb{L}^a_{\leq}$ and $\mathcal{B} \in \mathbb{L}^a_{\leq} \setminus \mathbb{L}$ also show that there is no containment between $\mathbb{L}$ and $\mathbb{L}^a_{\leq}$. We can thus sketch a map (Figure 1.2) of the terrain we will be living in throughout this thesis; the $\mathcal{P}_i$ and $\mathcal{Q}_j$ properties are introduced in the next section.

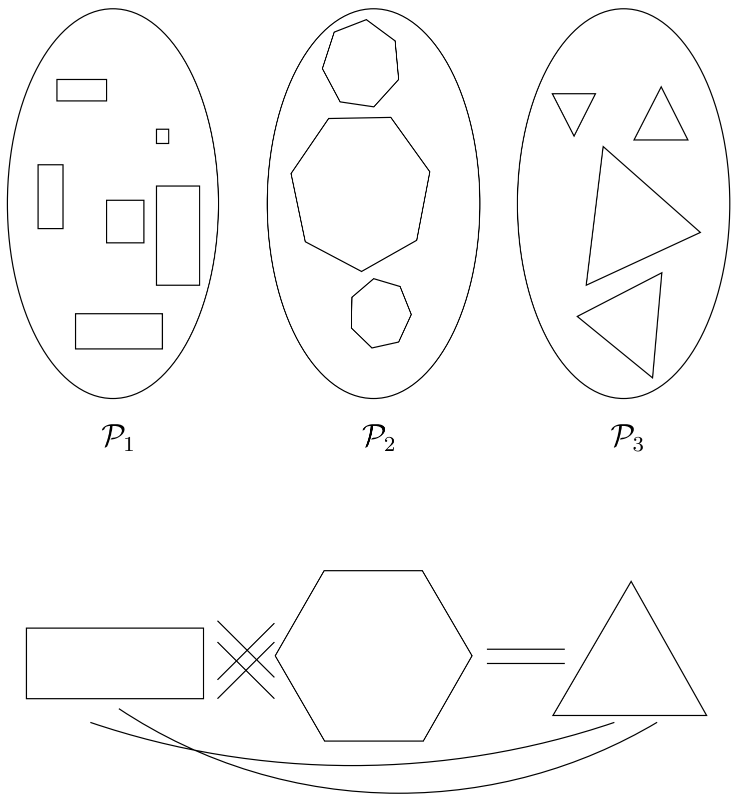


Figure 1.1: $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, and a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$-colourable graph.

A property is *reducible* if it is the product of at least two properties; otherwise it is *irreducible*. Let $\mathbb{P}$ be a class of properties closed under multiplication. Then $\mathcal{P} \in \mathbb{P}$ is *reducible over* $\mathbb{P}$ if it is the product of at least two properties from $\mathbb{P}$; otherwise it is *irreducible over* $\mathbb{P}$. Similarly, $\mathcal{P}$ is *uniquely factorisable over* $\mathbb{P}$ if it has only one factorisation into properties that are irreducible over $\mathbb{P}$; if $\mathbb{P}$ is the class $\mathbb{U}$ of all properties, we just say that $\mathcal{P}$ is uniquely factorisable.

Any irreducible property that happens to be in $\mathbb{P}$ is clearly irreducible over $\mathbb{P}$, and for the classes in $\{\mathbb{L}, \mathbb{L}_{\leq}, \mathbb{L}^a, \mathbb{L}^a_{\leq}\}$ we show (Theorem 5.4.1) that the converse holds, partly by using a result of Broere and Bucko [18] on the existence of uniquely colourable graphs.

Note that every additive induced-hereditary property contains $\mathcal{O}$, and thus every property reducible over $\mathbb{L}^a_{\leq}$ contains $\mathcal{O}^2$, the set of bipartite graphs.

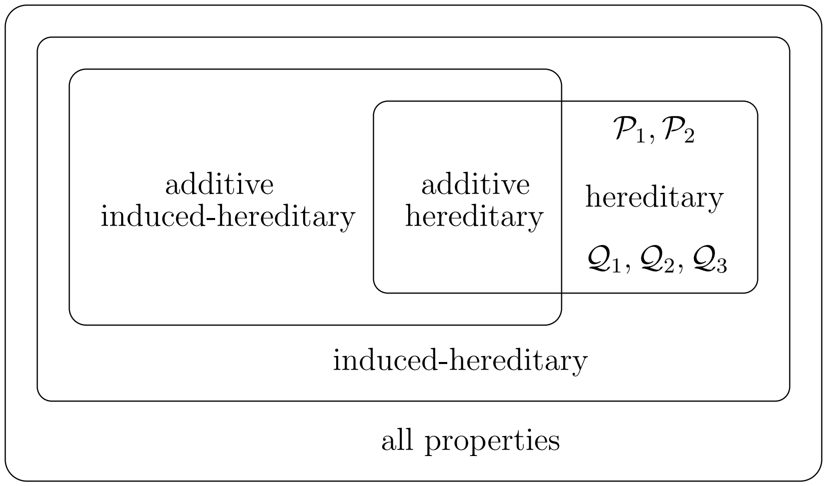


Figure 1.2: Various classes of properties.

As an easy exercise, let $\mathcal{P}$ be some fixed property such that $\mathcal{P}^i \subsetneqq \mathcal{P}^{i+1}$ for all $i \geq 1$, and define $\mathbb{P} := \{\mathcal{P}^2, \mathcal{P}^3, \ldots\}$. The only properties that are irreducible over $\mathbb{P}$ are $\mathcal{P}^2$ and $\mathcal{P}^3$, and the only ones that are uniquely factorisable over $\mathbb{P}$ are $\mathcal{P}^i, i = 2, 3, 4, 5, 7$.

We always have $\mathcal{P}^i \subseteq \mathcal{P}^{i+1}$, and equality can hold, for example, when $\mathcal{P} = \{G \mid |V(G)| \geq 10\}$. However, if $\mathcal{P}$ is induced-hereditary, then $K_1 \in \mathcal{P}$ and so we have strict containment — let H be a graph not in $\mathcal{P}^i$, with the least possible number of vertices, and let v be in $V(H)$; then because $H - v \in \mathcal{P}^i$, and $K_1 \in \mathcal{P}$, H is in $\mathcal{P}^{i+1}$.

1.3 Unique factorisation

Irreducible properties are analogous to prime numbers, and it is therefore natural to ask whether graph properties factor uniquely into irreducible properties. Semanišin [58] showed how to construct simple counterexamples that are even hereditary, such as:

$$\{K_1\}^4 = \{G \mid |V(G)| \leq 4\} = \{K_1\}^2 \circ \{K_2, \overline{K}_2\}.$$

Note that $\{K_2, \overline{K}_2\}$ is irreducible, as otherwise its factors would have to be $\{K_1\}$ and $\{K_1\}$, but $K_1 \in \{K_1\}^2$. We use similar examples in Theorem 5.2.1

to determine the (few) induced-hereditary properties that are uniquely factorisable.

It is particularly interesting that we have a hereditary property with two different factorisations, where even the number of irreducible factors is different. One of these factors, however, is not induced-hereditary; one might hope that properties in $\mathbb{L}_{\leq}$ would be uniquely factorisable over $\mathbb{L}_{\leq}$, or at least [43, Section 17.9], that properties in $\mathbb{L}$ are uniquely factorisable over $\mathbb{L}$. However, Mihók, Semanišin and Vasky [51, Example 4.2] found distinct irreducible hereditary properties $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$, such that $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \mathcal{Q}_3$; note that here too, the number of irreducible factors is different.

The same authors also gave a factorisation for *additive* hereditary properties (over $\mathbb{L}^a$) and stated that it was unique, though we find their uniqueness proof unsatisfactory, as we explain in Section 3.2. In [52], Mihók gave a remarkably general construction of what he called uniquely $\mathcal{P}$-decomposable graphs, and used this to produce a factorisation for all of $\mathbb{L}^a_{\leq}$; this was claimed to be unique using essentially the same proof as in [51]. As a major contribution of this thesis, we prove both uniqueness claims; these are also essential in proving the other main result, discussed in the next section. We actually state and prove our uniqueness results for the wider 'compositive' classes of properties, which we introduce in the next chapter:

Theorems 3.3.3, 4.3.7, 5.4.2. *Additive induced-hereditary properties are uniquely factorisable into irreducible additive induced-hereditary properties. In fact, for each* $\mathbb{P} \in \{\mathbb{L}^c, \mathbb{L}^a, \mathbb{L}^a_{\leq}, \mathbb{L}^{dc}_{\leq}\}$*, any property in the class* $\mathbb{P}$ *is uniquely factorisable over* $\mathbb{P}$*.* □

We note that, earlier on, unique factorisation over $\mathbb{L}$ was established correctly [47] for *hom-properties* (the class $\mathbb{H} := \{\rightarrow H \mid H$ a finite, simple graph$\}$), a significant subclass of $\mathbb{L}^a$ which is also closed under multiplication.

After we showed our results to Mihók, he came up with a simpler proof of the most crucial part, which appears in [6]. In that paper, the results are stated for directed edge-coloured hypergraphs, as the proofs carry over with only minor changes (the definition of $*$-join, and the precise wording of the proof of Theorem 4.2.2, are modified to reflect the use of hyperedges). Readers interested in such a generalisation to different structures can check that almost all the proofs carry through word for word.

1.4 Complexity

The computational complexity of deciding whether an arbitrary graph is $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable has attracted attention from various authors; we are interested here in fixed additive induced-hereditary properties $\mathcal{P}_1, \ldots, \mathcal{P}_n$, $n \geq 2$ (we discuss these restrictions at the end of the section). We call this problem $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-COLOURING,$(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-PARTITIONING or $(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n)$-RECOGNITION.

This problem can be arbitrarily hard — Scheinerman [57] showed that for any computational problem Π, there is an additive induced-hereditary property $\mathcal{P}$ whose recognition is polynomially equivalent to Π. (Problems Π_1 and Π_2 are *polynomially equivalent* if there are polynomial transformations from Π_1 to Π_2 and from Π_2 to Π_1.)

Brown [22] noted that in fact, since there are countably many algorithms, but uncountably many sets of finite connected graphs, each one characterising a different additive hereditary property $\mathcal{P}$ (see Proposition 2.1.1, Theorem 2.3.2(E)), $\mathcal{P}$-recognition will be undecidable with probability 1.

Recall that every additive induced-hereditary property $\mathcal{P}_i$ contains $\mathcal{O}$, and the case $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{O}$, $n \geq 3$, is one of the earliest and best known NP-complete problems [44]. For $n = 2$ this particular problem has a well-known polynomial-time algorithm, but Kratochvíl and Schiermeyer [48] conjectured that for any reducible additive (induced-)hereditary property $\mathcal{P} \neq \mathcal{O}^2$, $\mathcal{P}$-RECOGNITION is NP-hard.

The celebrated Hell-Nešetřil result [42] showed that hom-properties are NP-complete to recognise, even if they are irreducible, with the exception of $\mathcal{O}$ and $\mathcal{O}^2$.

Achlioptas [1] used uniquely colourable graphs to give an easy reduction (Theorem 6.1.1) from $\mathcal{P}^n$-COLOURING to the usual n-COLOURING of graphs, for irreducible additive induced-hereditary $\mathcal{P}$, thus settling the conjecture for this important class of properties when $n \geq 3$. He also gave a neat, but more involved proof, that $\mathcal{P}^2$-COLOURING is also NP-hard, for irreducible $\mathcal{P} \neq \mathcal{O}$.

Various authors before him had proved special cases of Achlioptas' result. Brown and Corneil [21, 23] showed that $\mathcal{P}^k$-RECOGNITION is NP-hard when $\mathcal{P}$ is the set of perfect graphs and $k \geq 2$, while Hakimi and Schmeichel [41] did the case $\{\text{forests}\}^2$. There was particular interest in G-free k-COLOURING, that is, $\mathcal{P}^k$-COLOURING where $\mathcal{P} = \{H \mid G \not\leq H\}$, for fixed G and k. Graph colouring is actually a particular case of this problem ($G = K_2$), while subchromatic number [3, 35] (partitioning into subgraphs whose components are all cliques) is the case $G = P_3$. Brown [22] proved the case where G is 2-connected, and then Achlioptas [1] showed NP-completeness for all G, though his proof works equally well for $\mathcal{P}^k$-COLOURING, so long as $\mathcal{P}$ is irreducible.

Our first complexity result, Theorem 6.1.2, is that $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is at least as hard as $\mathcal{P}$- or $\mathcal{Q}$-RECOGNITION, if $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary. This automatically extends Achlioptas' result to $\mathcal{P}^k$-COLOURING even for reducible $\mathcal{P}$. In fact, this already settles the Kratochvíl-Schiermeyer conjecture for any property divisible by $\mathcal{P}^2$, $\mathcal{P} \neq \mathcal{O}$.

Some work had been done on products of distinct properties. Monien (see [17]) pointed out that a proof of Garey et al. [37] essentially showed $(\mathcal{O}, \{\text{forests}\})$-COLOURING to be NP-complete, while Brandstädt et al. [17, Thm. 3] proved the case $(\mathcal{O}, \{P_4, C_4\} - \text{free graphs})$. Kratochvíl and Schiermeyer [48] proved NP-hardness for every property of the form $\mathcal{O} \circ \mathcal{Q}$, where $\mathcal{Q} \neq \mathcal{O}$ is additive hereditary.

The Kratochvíl-Schiermeyer proof has two parts, which can be extended to cover the case $\mathcal{P} \circ \mathcal{Q}$ for *any* additive hereditary $\mathcal{P}$ and $\mathcal{Q}$, settling their original conjecture. However, and surprisingly, the *easier* part of their proof alone can be adapted to extend this result even to additive induced-hereditary $\mathcal{P}$ and $\mathcal{Q}$, which we do in our second main contribution:

Theorem 6.1.4. *If $\mathcal{P}$ and $\mathcal{Q}$ are additive induced-hereditary graph families, then $(\mathcal{P}, \mathcal{Q})$-COLOURING is NP-hard, with the sole exception of graph 2-COLOURING (the case $\mathcal{P} = \mathcal{Q} = \mathcal{O}$); with the same exception, $(\mathcal{P}, \mathcal{Q})$-COLOURING is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both in NP.* □

Problems such as the following (for an arbitrary graph G) are therefore now known to be NP-complete. Can $V(G)$ be partitioned into $A \cup B$, so that $G[A]$ is a line-graph and $G[B]$ is a forest? Can G be partitioned into a planar graph and a perfect graph? For fixed k, ℓ, m, can G be partitioned into a subgraph of girth at least k, a subgraph of maximum degree at most ℓ, and an m-edge-colourable subgraph?

We now discuss our reasons for restricting ourselves to fixed reducible additive induced-hereditary properties.

One of the usual graph colouring problems is to ask for the *minimum* k such that a graph G is k-colourable, rather than specifying an integer r and asking whether G is r-colourable. In a generalised colouring context, we could fix a (reducible or irreducible) property $\mathcal{P}$ and ask for the minimum k such that $G \in \mathcal{P}^k$. This problem, however, cannot be easier than if we fix k as well, so if $\mathcal{P}$ is additive induced-hereditary, the "minimum k" problem is NP-hard.

We focus almost exclusively on induced-hereditary properties throughout the whole thesis. The more specific focus here on additive induced-hereditary properties is motivated partly by our proof techniques, and partly by the fact that the complexity results simply do not hold for wider classes of properties (assuming P $\neq$ NP), as many non-additive properties are finite (consider $H_{\subseteq}$ or $H_{\leq}$, for example, which are compositive but not additive).

Even the intuitively obvious result, that $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is at least as hard as $\mathcal{P}$-RECOGNITION, does not hold for all additive $\mathcal{Q}$ (note that $\mathcal{Q} := \{G \mid |V(G)| \geq 10\}$ is additive, and that, for all $\mathcal{P}$, $(\mathcal{P} \circ \mathcal{Q}) \supseteq \mathcal{Q}$). By contrast, if $\mathcal{P}$ and $\mathcal{Q}$ are both additive induced-hereditary, then we have not only this nice "monotonic complexity", but also the pleasing feature that $(\mathcal{P}, \mathcal{Q})$-colourings of each component of a graph G together give us a colouring of G, while a $(\mathcal{P}, \mathcal{Q})$-colouring of G induces obvious colourings of all its induced-subgraphs.

There is a distinction between the complexity of recognising reducible and irreducible properties (again, assuming P $\neq$ NP), something that was already clear from the results of Achlioptas [1] and the earlier results of Brown [22] — for example, recognition of K_3-free graphs is trivial, while that of $\{K_3 - \text{free graphs}\}^2$ is NP-complete.

Berger [8] has also proved that any reducible additive induced-hereditary property has infinitely many minimal forbidden subgraphs (Theorem 2.1.4), providing support for the complexity conjecture. If P $\neq$ NP, this would in fact follow from the NP-hardness of recognising reducible properties (although Berger proved a somewhat stronger result).

Finally, we note that Theorem 6.1.4 depends on the unique factorisation results, specifically on Theorem 4.3.3, that shows that Mihók's uniquely $\mathcal{P}$-decomposable graphs (Theorem 4.2.2, Corollary 4.2.4) are in fact uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable.

1.5 Outline of the thesis

We end our introduction with a brief summary of the subsequent chapters. We note that, if we discuss Chapters 3, 4 and 6 quite briefly here, this should not reflect on their relative importance, but only on the fact that we have already devoted a few pages to those chapters.

Chapter 2 gives some of the background material on hereditary and induced-hereditary properties, and introduces various forms of compositivity, generalisations of additivity that we will be considering in this thesis. We show how generating sets and minimal forbidden (induced) subgraphs have been used to characterise induced-hereditary and compositive properties, thus giving readers an opportunity to familiarise themselves with some of the basic tools we will use. In particular, we characterise the minimal forbidden subgraphs of a hereditary property in terms of its minimal forbidden induced-subgraphs, and *vice versa*.

Chapter 3 describes Mihók et al.'s canonical factorisation for the hereditary case, and explains the gap in their proof of uniqueness. We then give our own uniqueness proof.

Chapter 4 is the induced-hereditary analogue of Chapter 3, with substantially more technical detail. We prove unique factorisation, and generalise Mihók's construction of uniquely $\mathcal{P}$-decomposable graphs, for additive and induced-hereditary disjoint compositive properties.

Chapter 5 discusses factorisations involving properties that are not additive, or not induced-hereditary. We first explore the factorisation of complementary properties — both the setwise complement, and the set of graph complements — of additive induced-hereditary properties. Then we determine the few induced-hereditary properties which remain uniquely factorisable when the factors need not be induced-hereditary. We look at uniquely colourable graphs, and use them to show that a "greedy algorithm" will extract generating sets for $\mathcal{A}$ and $\mathcal{B}$ from a generating set for $\mathcal{A} \circ \mathcal{B}$. The existence of uniquely colourable graphs also shows that additive induced-hereditary properties that factor into two or more properties, also factor into two or more additive induced-hereditary properties; this, and similar results, concludes the unique factorisation proofs.

Chapter 6 gives a short proof that $(\mathcal{P}, \mathcal{Q})$-COLOURING is NP-hard, for any fixed additive induced-hereditary properties $\mathcal{P}$ and $\mathcal{Q}$.

Chapter 7 is devoted to open questions and directions for further research.

Chapter 2

Supergraphs and forbidden subgraphs

This chapter is intended to give the flavour of our approach to induced-hereditary properties, with many important results from other authors, along with our own results. In the first three sections we see that hereditary and induced-hereditary properties are precisely those that can be defined "from below" — by excluding a fixed list of subgraphs or induced-subgraphs; while the various classes of compositive properties that are the focus of attention in this thesis can be characterised "from above" — by taking all (finite) subgraphs of a fixed graph.

Section 2.3 contains the more difficult results that concern 'induced-hereditary disjoint compositive' properties. Results for additive induced-hereditary and, even more so, induced-hereditary disjoint compositive properties, often mirror the results for hereditary or additive hereditary properties, but are more difficult to prove.

In Section 2.4, we consider which of the characterisations hold when we allow infinite graphs; and in Section 2.5 we make a brief attempt at characterising compositive properties in terms of forbidden subgraphs.

2.1 Minimal forbidden structures — background

This section looks at sets of minimal forbidden subgraphs and induced-subgraphs. After seeing that they define hereditary and induced-hereditary properties, respectively, we show how the set of forbidden induced-subgraphs tells us whether a property is hereditary, and what its forbidden subgraphs are.

The material in this section is meant as background, though we note that Propositions 2.1.7–2.1.9 are original, as far as we know. None of the proofs given here are particularly difficult, and, apart from Propositions 2.1.1 and 2.1.3, we do not need the results in the rest of the thesis.

Let $\preceq$ be a partial order on a set $\mathcal{S}$. An element $s \in \mathcal{S}$ is a *$\preceq$-predecessor* of t if $s \preceq t$; if, moreover, $s \neq t$, then s is a *proper $\preceq$-predecessor* of t, and we write $s \prec t$. For any set $T \subseteq \mathcal{S}$, we define $\min_{\preceq}(T) := \{t \in T \mid \nexists s \in T, s \prec t\}$. A (finite or infinite) *chain* with respect to $\preceq$ is a sequence of distinct elements such that $s_1 \preceq s_2 \preceq \ldots$; while a (finite or infinite) descending chain is a sequence of distinct elements such that $s_1 \succeq s_2 \succeq \ldots$. An *antichain* (with respect to $\preceq$) is a set where no member is a $\preceq$-predecessor of another.

A *property* is a subset of $\mathcal{S}$. A property $\mathcal{P}$ is *$\preceq$-hereditary* if, whenever $s \preceq t$ and $t \in \mathcal{P}$, then also $s \in \mathcal{P}$; that is, whenever t is in $\mathcal{P}$, all its $\preceq$-predecessors are also in $\mathcal{P}$. A *minimal forbidden $\preceq$-predecessor* is an element $t \notin \mathcal{P}$ all of whose proper $\preceq$-predecessors (if any) are in $\mathcal{P}$; the set of all such elements is denoted by $\mathcal{F}_{\preceq}(\mathcal{P})$. As suggested by their name, the minimal forbidden $\preceq$-predecessors are the minimal elements, under the $\preceq$ order, among the elements not in $\mathcal{P}$: $\mathcal{F}_{\preceq}(\mathcal{P}) := \min_{\preceq}\{t \mid t \notin \mathcal{P}\}$.

$\mathcal{F}_{\preceq}(\mathcal{P})$ must definitely be an antichain with respect to $\preceq$, that is, no member of $\mathcal{F}_{\preceq}(\mathcal{P})$ is a $\preceq$-predecessor of another member of $\mathcal{F}_{\preceq}(\mathcal{P})$. Conversely, any set A that is an antichain under $\preceq$ is the set of *minimal* forbidden $\preceq$-predecessors of some $\preceq$-hereditary property, specifically $A = \mathcal{F}_{\preceq}(\mathcal{P})$ where $\mathcal{P} = \{s \in \mathcal{S} \mid \forall a \in A, a \npreceq s\}$.

A property is *characterised by its minimal forbidden $\preceq$-predecessors* when t is in $\mathcal{P}$ iff none of its $\preceq$-predecessors is in $\mathcal{F}_{\preceq}(\mathcal{P})$. Greenwell, Hemminger and Klerlein proved the following simple result:

2.1.1. Proposition [40]. *Let $\preceq$ be a partial order with no infinite descending chain. A property $\mathcal{P}$ is $\preceq$-hereditary if and only if it is characterised by its minimal forbidden $\preceq$-predecessors.*

Proof: ($\Leftarrow$) Let $s \preceq t \in \mathcal{P}$. Every $\preceq$-predecessor of s is also a $\preceq$-predecessor of t; since t is in $\mathcal{P}$, none of the $\preceq$-predecessors of s can be in $\mathcal{F}_{\preceq}(\mathcal{P})$, so s is also in $\mathcal{P}$.

($\Rightarrow$) Let $\mathcal{P}$ be $\preceq$-hereditary. If an element s is in $\mathcal{P}$, then all its $\preceq$-predecessors are in $\mathcal{P}$; in particular, none of them is in $\mathcal{F}_{\preceq}(\mathcal{P})$. Conversely, suppose s is not in $\mathcal{P}$. If all its proper $\preceq$-predecessors are in $\mathcal{P}$, then $s \in \mathcal{F}_{\preceq}(\mathcal{P})$. Otherwise, let $s_1 \notin \mathcal{P}$ be a $\preceq$-predecessor of s; either $s_1 \in \mathcal{F}_{\preceq}(\mathcal{P})$, or we can find an element $s_2 \preceq s_1$ that is not in $\mathcal{P}$. Continuing in this manner, we must find $s_r \preceq \cdots \preceq s_1 \preceq s$ such that $s_r \in \mathcal{F}_{\preceq}(\mathcal{P})$, for if not, we would have an infinite descending $\preceq$-chain starting from s. $\square$

The restriction on descending chains cannot be removed. Consider, for example, the "supergraph" partial order, denoted by $\supseteq$ (from now on, $\mathcal{S}$ will always be the set of finite simple graphs). The property $\mathcal{P}$ of having an edge is $\supseteq$-hereditary, but none of the graphs outside $\mathcal{P}$ are *minimal* forbidden $\supseteq$-predecessors: $\overline{K}_r$ always has proper $\supseteq$-predecessors (such as $\overline{K}_{r+1}$) that are not in $\mathcal{P}$. Thus $\mathcal{F}_{\supseteq}(\mathcal{P}) = \emptyset$, so it definitely does not characterise $\mathcal{P}$.

For which graph properties $\mathcal{P}$ is it true that the additive hereditary property generated by $\mathcal{P}$ is the same as what we obtain by first generating an additive property and then generating a hereditary property from that, or vice versa? We give an easy necessary condition below; similar results can be found in [13, 45] and [11, Section 4].

The *$\preceq$-hereditary property generated by* $\mathcal{P}$ is the smallest such property that contains $\mathcal{P}$, denoted $\mathcal{P}_{\preceq}$; this is easily seen to be the intersection of all $\preceq$-hereditary properties that contain $\mathcal{P}$. Similarly $\mathcal{P}^a$ is the smallest additive property containing $\mathcal{P}$. The smallest additive $\preceq$-hereditary property containing $\mathcal{P}$ is denoted by $\mathcal{P}^a_{\preceq}$.

2.1.2. Proposition. *Let $\mathcal{P}$ be an arbitrary property, and $\preceq$ a partial order on (finite, simple) graphs. Then*

$$\begin{array}{lrcl} & \mathcal{P}_{\preceq} & = & \{G \mid \exists H \in \mathcal{P}, G \preceq H\} \\ \textit{and} & \mathcal{P}^a & = & \{H_1 \cup \cdots \cup H_r \mid H_1, \ldots, H_r \in \mathcal{P}\}. \\ \textit{Thus} & (\mathcal{P}^a)_{\preceq} & = & \{G \mid \exists H_1, \ldots, H_r \in \mathcal{P}, G \preceq H_1 \cup \cdots \cup H_r\} \\ \textit{and} & (\mathcal{P}_{\preceq})^a & = & \{G_1 \cup \cdots \cup G_r \mid \exists H_1, \ldots, H_r \in \mathcal{P}, \forall i, G_i \preceq H_i\}. \end{array}$$

If $\preceq$ satisfies (A): $[G_1 \preceq H_1, G_2 \preceq H_2] \Rightarrow [G_1 \cup G_2 \preceq H_1 \cup H_2]$, then $\mathcal{P}^a_{\preceq} = (\mathcal{P}^a)_{\preceq}$.
If $\preceq$ satisfies (B): $[G \preceq H_1 \cup H_2] \Rightarrow [\exists G_i \preceq H_i, G = G_1 \cup G_2]$, then $\mathcal{P}^a_{\preceq} = (\mathcal{P}_{\preceq})^a$.

Proof: $\mathcal{P}_{\preceq}$ must contain all $\preceq$-predecessors of graphs in $\mathcal{P}$, and the set obtained this way is $\preceq$-hereditary (because $\preceq$ is transitive); $\mathcal{P}^a$ must contain all disjoint unions of graphs in $\mathcal{P}$, and the set obtained this way is additive.

Since $\mathcal{P}^a_{\preceq}$ is additive, $\mathcal{P}^a \subseteq \mathcal{P}^a_{\preceq}$, and since $\mathcal{P}^a_{\preceq}$ is $\preceq$-hereditary, $(\mathcal{P}^a)_{\preceq} \subseteq \mathcal{P}^a_{\preceq}$. Now $(\mathcal{P}^a)_{\preceq}$ is $\preceq$-hereditary by definition, and we claim that if condition (A) holds, then it is also additive. Suppose G_1 and G_2 are in $(\mathcal{P}^a)_{\preceq}$; then $G_i \preceq H_i \in \mathcal{P}^a$, and by (A) we have $G_1 \cup G_2 \preceq H_1 \cup H_2$. Clearly $H_1 \cup H_2$ is in $\mathcal{P}^a$, so $G_1 \cup G_2$ is in $(\mathcal{P}^a)_{\preceq}$.

Since $\mathcal{P}^a_{\preceq}$ is $\preceq$-hereditary, $\mathcal{P}_{\preceq} \subseteq \mathcal{P}^a_{\preceq}$, and since $\mathcal{P}^a_{\preceq}$ is additive, $(\mathcal{P}_{\preceq})^a \subseteq \mathcal{P}^a_{\preceq}$. Now $(\mathcal{P}_{\preceq})^a$ is additive by definition, and we claim that if condition (B) holds, then it is also $\preceq$-hereditary. Suppose $G \preceq H \in (\mathcal{P}_{\preceq})^a$. Then $H = H_1 \cup \cdots \cup H_r$, where each H_i is in $\mathcal{P}_{\preceq}$, and by repeated use of (B), $G = G_1 \cup \cdots \cup G_r$ where, for each i, $G_i \preceq H_i$ and therefore $G_i \in \mathcal{P}_{\preceq}$. Thus $G \in (\mathcal{P}_{\preceq})^a$. □

Hereditary and induced-hereditary properties are $\subseteq$-hereditary and $\leq$-hereditary, respectively; while subgraphs and induced-subgraphs are the $\subseteq$- or $\leq$-predecessors. Since the subgraph and induced-subgraph partial orders satisfy both condition (A) and (B) in Proposition 2.1.2, we have $\mathcal{P}^a_{\subseteq} = (P_{\subseteq})^a = (\mathcal{P}^a)_{\subseteq}$ and $\mathcal{P}^a_{\leq} = (P_{\leq})^a = (\mathcal{P}^a)_{\leq}$. These two partial orders are also, in some sense, distributive over property multiplication, as we show next (see also Theorem 5.4.1).

2.1.3. Proposition. *Let $\mathcal{P}$ and $\mathcal{Q}$ be any two graph properties.*

A. $\mathcal{P}_{\subseteq} \circ \mathcal{Q}_{\subseteq} = (\mathcal{P} \circ \mathcal{Q})_{\subseteq}$.

B. $\mathcal{P}_{\leq} \circ \mathcal{Q}_{\leq} = (\mathcal{P} \circ \mathcal{Q})_{\leq}$.

Thus a hereditary (resp. induced-hereditary) property is irreducible iff it is irreducible over $\mathbb{L}$ (resp. $\mathbb{L}_{\leq}$).

Proof: Since $\mathcal{P}_{\subseteq}$ and $\mathcal{Q}_{\subseteq}$ are hereditary, their product is also hereditary; so $\mathcal{P}_{\subseteq} \circ \mathcal{Q}_{\subseteq}$ is a hereditary property that contains $\mathcal{P} \circ \mathcal{Q}$, and thus contains $(\mathcal{P} \circ \mathcal{Q})_{\subseteq}$. Conversely, any $G \in \mathcal{P}_{\subseteq} \circ \mathcal{Q}_{\subseteq}$ is a subgraph of some $H \in \mathcal{P} \circ \mathcal{Q} \subseteq (\mathcal{P} \circ \mathcal{Q})_{\subseteq}$; thus G is in $(\mathcal{P} \circ \mathcal{Q})_{\subseteq}$.

If $\mathcal{P} \in \mathbb{L}$ is irreducible, then trivially it is irreducible over $\mathbb{L}$. So suppose $\mathcal{P}$ is reducible, say $\mathcal{P} = \mathcal{Q} \circ \mathcal{R}$, where $\mathcal{Q}$ and $\mathcal{R}$ need not be hereditary. Then $\mathcal{P} = \mathcal{Q} \circ \mathcal{R} \subseteq \mathcal{Q}_{\subseteq} \circ \mathcal{R}_{\subseteq} = (\mathcal{Q} \circ \mathcal{R})_{\subseteq} = \mathcal{P}_{\subseteq} = \mathcal{P}$, so we have equality throughout; thus $\mathcal{P}$ is the product of $\mathcal{Q}_{\subseteq}$ and $\mathcal{R}_{\subseteq}$, which are both in $\mathbb{L}$.

The induced-hereditary parts are proved similarly. □

As noted by Brown [22, Section 3], if S is a set of cycles, then S is a $\leq$-antichain, so there is an induced-hereditary property $\mathcal{P}_S$ with $\mathcal{F}_{\leq}(\mathcal{P}_S) = S$. There are uncountably many such sets, each of them characterising a different induced-hereditary property. It will follow from Theorem 2.3.2(E) that these uncountably many properties are all additive hereditary. Interestingly, whenever S is finite, $\mathcal{P}_S$ is irreducible, as shown by Berger:

2.1.4. Theorem [8]**.** *If $\mathcal{P}$ is a reducible property in $\mathbb{L}^a_{\leq}$, then the set of blocks of graphs in $\mathcal{F}_{\leq}(\mathcal{P})$ is infinite, and hence $\mathcal{F}_{\leq}(\mathcal{P})$ is infinite.* □

Every hereditary property is induced-hereditary, so it is characterised by both $\mathcal{F}_{\subseteq}(\mathcal{P})$ and $\mathcal{F}_{\leq}(\mathcal{P})$. For example, for forests the two sets both consist of all cycles, while for bipartite graphs both sets are the collection of odd cycles. When $\mathcal{P}$ is the set of graphs with at most k vertices, $\mathcal{F}_{\subseteq}(\mathcal{P}) = \{\overline{K}_{k+1}\}$, while $\mathcal{F}_{\leq}(\mathcal{P})$ contains all the graphs on $k+1$ vertices. When $\mathcal{P}$ is the set of graphs with at most k vertices in each component, $\mathcal{F}_{\subseteq}(\mathcal{P})$ consists of all trees on $k+1$ vertices, while $\mathcal{F}_{\leq}(\mathcal{P})$ is the set of all connected graphs on $k+1$ vertices.

In general, how do we obtain $\mathcal{F}_{\subseteq}(\mathcal{P})$ from $\mathcal{F}_{\leq}(\mathcal{P})$, or vice-versa? When are the two sets equal? If we are given an induced-hereditary property, can we recognise from $\mathcal{F}_{\leq}(\mathcal{P})$ whether $\mathcal{P}$ is actually hereditary? It is useful to start by writing the definitions of $\mathcal{F}_{\subseteq}(\mathcal{P})$ and $\mathcal{F}_{\leq}(\mathcal{P})$ explicitly:

$$\begin{array}{lll} G \in \mathcal{F}_{\subseteq}(\mathcal{P}) & \text{iff} & G \notin \mathcal{P} \text{ but } \forall v \in V(G), G - v \in \mathcal{P}, \\ & & \quad \text{and } \forall e \in E(G), G - e \in \mathcal{P} \\ G \in \mathcal{F}_{\leq}(\mathcal{P}) & \text{iff} & G \notin \mathcal{P} \text{ but } \forall v \in V(G), G - v \in \mathcal{P} \end{array}$$

This immediately tells us that $\mathcal{F}_{\subseteq}(\mathcal{P}) \subseteq \mathcal{F}_{\leq}(\mathcal{P})$.

$\mathcal{F}_{\leq}(\mathcal{P})$ must be an antichain with respect to $\leq$, but not necessarily with respect to $\subseteq$ — recall that when $\mathcal{P}$ is the set of graphs with at most k vertices, $\mathcal{F}_{\leq}(\mathcal{P})$ contains all the graphs on $k+1$ vertices. In that example, all graphs in $\mathcal{F}_{\leq}(\mathcal{P})$ happen to have the same order, but even when a hereditary property has minimal forbidden induced-subgraphs with different numbers of vertices, only the ones with equal order can be subgraphs of one another:

2.1.5. Lemma. *Let $\mathcal{P}$ be hereditary, with G and H both in $\mathcal{F}_{\leq}(\mathcal{P})$. If $G \subseteq H$, then $|V(G)| = |V(H)|$.*

Proof: If there is a vertex $v \in V(H) \setminus V(G)$, then $G \subseteq H - v$. Since

$H \in \mathcal{F}_{\leq}(\mathcal{P})$, $H - v \in \mathcal{P}$; and since $\mathcal{P}$ is hereditary, $G \in \mathcal{P}$, a contradiction. □

2.1.6. Lemma. *Let $\mathcal{P}$ be a hereditary graph property. If $H \in \mathcal{F}_{\leq}(\mathcal{P})$, then H has a spanning subgraph G that is in $\mathcal{F}_{\subseteq}(\mathcal{P})$.*

Proof: Since H is in $\mathcal{F}_{\leq}(\mathcal{P})$, it is not in $\mathcal{P}$; it must therefore contain a subgraph G in $\mathcal{F}_{\subseteq}(\mathcal{P})$. Since $\mathcal{F}_{\subseteq}(\mathcal{P}) \subseteq \mathcal{F}_{\leq}(\mathcal{P})$, G is in $\mathcal{F}_{\leq}(\mathcal{P})$, and by the previous lemma we have $|V(G)| = |V(H)|$. □

The graph $G \in \mathcal{F}_{\subseteq}(\mathcal{P})$, whose existence is guaranteed by the previous lemma, is also in $\mathcal{F}_{\leq}(\mathcal{P})$. Thus $\mathcal{F}_{\subseteq}(\mathcal{P})$ is the set of $\subseteq$-minimal elements of $\mathcal{F}_{\leq}(\mathcal{P})$. Moreover, since the graph G of Lemma 2.1.6 must be a *spanning* subgraph of H, we only need to look at graphs on the same number of vertices to decide if H is a $\subseteq$-minimal element. That is, if we define $\mathcal{F}_{\leq}(\mathcal{P}, n) := \{G \in \mathcal{F}_{\leq}(\mathcal{P}) \mid |V(G)| = n\}$, then

$$\mathcal{F}_{\subseteq}(\mathcal{P}) = \min_{\subseteq}(\mathcal{F}_{\leq}(\mathcal{P})) = \bigcup_{n} \min_{\subseteq}(\mathcal{F}_{\leq}(\mathcal{P}, n)).$$

Two results follow from this discussion:

2.1.7. Proposition. *Let $\mathcal{P}$ be hereditary. Then $\mathcal{F}_{\subseteq}(\mathcal{P})$ is finite if and only if $\mathcal{F}_{\leq}(\mathcal{P})$ is finite. $\mathcal{F}_{\subseteq}(\mathcal{P}) = \mathcal{F}_{\leq}(\mathcal{P})$ if and only if $\mathcal{F}_{\leq}(\mathcal{P})$ is an antichain under $\subseteq$, if and only if (for each $n \in \mathbb{N}$) $\mathcal{F}_{\leq}(\mathcal{P}, n)$ is an antichain under $\subseteq$.* □

Lemma 2.1.6 shows that the graphs of $\mathcal{F}_{\leq}(\mathcal{P})$ are all of the form $H + e_1 + \cdots + e_r$, where H is in $\mathcal{F}_{\subseteq}(\mathcal{P})$, and each e_i is an edge not in H. So, apart from the original definition

$$\mathcal{F}_{\leq}(\mathcal{P}) = \min_{\leq}\{G \mid G \notin \mathcal{P}\} = \min_{\leq}\{G \mid \exists H \in \mathcal{F}_{\subseteq}(\mathcal{P}), H \subseteq G\},$$

there is another way of obtaining $\mathcal{F}_{\leq}(\mathcal{P})$ from $\mathcal{F}_{\subseteq}(\mathcal{P})$:

$$\mathcal{F}_{\leq}(\mathcal{P}) = \min_{\leq}\{H + e_1 + \cdots + e_r \mid H \in \mathcal{F}_{\subseteq}(\mathcal{P})\}.$$

This offers a clearly finite method for obtaining $\mathcal{F}_{\leq}(\mathcal{P})$ when $\mathcal{F}_{\subseteq}(\mathcal{P})$ is finite, and also lets us characterise when $\mathcal{F}_{\leq}(\mathcal{P})$ and $\mathcal{F}_{\subseteq}(\mathcal{P})$ are equal:

2.1.8. Proposition. *Let $\mathcal{P}$ be a hereditary graph property. Then $\mathcal{F}_{\subseteq}(\mathcal{P}) = \mathcal{F}_{\leq}(\mathcal{P})$ if and only if, for all $H \in \mathcal{F}_{\subseteq}(\mathcal{P})$, and for all $e \notin E(H)$, there is some graph $G \in \mathcal{F}_{\subseteq}(\mathcal{P})$ such that $G \leq H + e$.*

Proof: Suppose that $\mathcal{F}_{\subseteq}(\mathcal{P}) = \mathcal{F}_{\leq}(\mathcal{P})$ for the hereditary property $\mathcal{P}$. If H is in $\mathcal{F}_{\subseteq}(\mathcal{P})$, then $H \notin \mathcal{P}$, so $H + e \notin \mathcal{P}$. Thus $H + e$ contains some induced-subgraph $G \in \mathcal{F}_{\leq}(\mathcal{P}) = \mathcal{F}_{\subseteq}(\mathcal{P})$.

For the other direction, we note that if every graph of the form $H + e_1 + \cdots + e_r$ (where H is in $\mathcal{F}_{\subseteq}(\mathcal{P})$, and $r \geq 0$) contains some $G \in \mathcal{F}_{\subseteq}(\mathcal{P})$ as an induced subgraph, then $\mathcal{F}_{\leq}(\mathcal{P}) = \mathcal{F}_{\subseteq}(\mathcal{P})$ by the preceding expression for $\mathcal{F}_{\leq}(\mathcal{P})$. The case $r = 0$ is trivial, and the case $r = 1$ follows by hypothesis, so suppose we have shown that every graph of the form $H+e_1+\cdots+e_t$ contains an induced copy of some graph from $\mathcal{F}_{\subseteq}(\mathcal{P})$; we want to show this is true also for $H+e_1+\cdots+e_t+e_{t+1}$. Let $G \leq H+e_1+\cdots+e_t$, where $G \in \mathcal{F}_{\subseteq}(\mathcal{P})$. If one of the end-vertices of e_{t+1} is not in $V(G)$, then $G \leq H + e_1 + \cdots + e_t + e_{t+1}$. Otherwise, $G + e_{t+1} \leq H + e_1 + \cdots + e_t + e_{t+1}$; but by hypothesis there is some $G' \in \mathcal{F}_{\subseteq}(\mathcal{P})$ such that $G' \leq G + e_{t+1}$, which proves the result. $\square$

There is a deceptively similar result that characterises those induced-hereditary properties that happen to be hereditary.

2.1.9. Proposition. *Let $\mathcal{P}$ be an induced-hereditary property. Then $\mathcal{P}$ is hereditary if and only if, for all $H \in \mathcal{F}_{\leq}(\mathcal{P})$, and for all $e \notin E(H)$, there is some graph $G \in \mathcal{F}_{\leq}(\mathcal{P})$ such that $G \leq H + e$. In this case, $\mathcal{P} = \{F \mid \forall G \in \mathcal{F}_{\leq}(\mathcal{P}), G \nsubseteq F\}$.*

Proof: If $\mathcal{P}$ is hereditary, and H is in $\mathcal{F}_{\leq}(\mathcal{P})$, then $H \notin \mathcal{P}$, so $H + e \notin \mathcal{P}$. Thus $H + e$ contains some induced-subgraph $G \in \mathcal{F}_{\leq}(\mathcal{P})$.

For the converse, note that $\mathcal{Q} := \{F \mid \forall G \in \mathcal{F}_{\leq}(\mathcal{P}), G \nsubseteq F\}$ is clearly hereditary, so we need to show that $\mathcal{P} = \mathcal{Q}$. If F is in $\mathcal{Q}$, it does not contain any graph in $\mathcal{F}_{\leq}(\mathcal{P})$ as a subgraph (or, in particular, as an *induced*-subgraph), so it is also in $\mathcal{P}$. Now suppose F is not in $\mathcal{Q}$, that is, it contains some graph in $\mathcal{F}_{\leq}(\mathcal{P})$ as a subgraph. Among the graphs in $\mathcal{F}_{\leq}(\mathcal{P})$ contained in F as a subgraph, let H have the least number n of vertices, and among all those on n vertices, let H have the largest possible number m of edges.

Suppose H is not an induced subgraph of F, that is, there is some edge $vw \in E(F) \setminus E(H)$, where v and w are both in $V(H)$. By hypothesis, there is some graph $G \in \mathcal{F}_{\leq}(\mathcal{P})$ such that $G \leq H + vw$. In particular, G is a graph in $\mathcal{F}_{\leq}(\mathcal{P})$ that is contained in F as a subgraph. By choice of H, G must have n vertices, but this means that G has $m+1$ edges, a contradiction. So H must be an *induced*-subgraph of F, and so $F \notin \mathcal{P}$. $\square$

2.2 Compositivity

We introduce some natural extensions of additivity in this section, and give some characterisations of these concepts. In the next chapter we will prove unique factorisation not just for additive hereditary and additive induced-hereditary properties, but for the wider classes of properties that we consider here.

In an additive property $\mathcal{P}$, for any two graphs $G_1, G_2 \in \mathcal{P}$, the graph $G_1 \cup G_2$ is also in $\mathcal{P}$. Thus the set $\mathcal{K}$ of cliques is not additive, but by adding (all) edges between G_1 and G_2 in $\mathcal{K}$ we get another graph in $\mathcal{K}$. If we fix a graph H and consider the set $H_{\leq}$ of its induced-subgraphs, adding edges between $G_1, G_2 \in H_{\leq}$ need not give us another graph in $H_{\leq}$; but if we are also allowed to identify some vertices of G_1 and G_2 we can produce a graph in $H_{\leq}$. With this motivation, we define new classes of properties as follows.

A *(vertex-disjoint)* $\leq$*-composition* of graphs G_1 and G_2 is a graph H that contains G_1 and G_2 as (vertex-disjoint) induced-subgraphs. A property $\mathcal{P}$ is *induced-hereditary (disjoint) compositive* if it is induced-hereditary, and any pair of graphs $G_1, G_2 \in \mathcal{P}$ has a (vertex-disjoint) $\leq$-composition in $\mathcal{P}$.

Similarly, we can define *(vertex-disjoint)* $\subseteq$*-compositions*, and *hereditary (disjoint) compositive* properties. Induced-hereditary compositivity was introduced by Scheinerman [55] in his study of intersection properties. The other classes of properties seem to be new.

Recall that the classes of hereditary and additive hereditary properties are $\mathbb{L}$ and $\mathbb{L}^a$, respectively; the induced-hereditary analogues are $\mathbb{L}_{\leq}$ and $\mathbb{L}^a_{\leq}$. Similarly, we will use $\mathbb{L}^c$ and $\mathbb{L}^{dc}$ for the classes of hereditary compositive and hereditary disjoint compositive properties, respectively, and $\mathbb{L}^c_{\leq}$ and $\mathbb{L}^{dc}_{\leq}$ for the induced-hereditary analogues.

Some observations are in order.

- Compositivity, unlike additivity, is not defined by itself — in a hereditary compositive property we want $G_i \subseteq H$, whereas in an induced-hereditary compositive property we want $G_i \leq H$, $i = 1, 2$. This is why we write the terms 'compositive' and 'disjoint compositive' after the qualifiers 'hereditary' and 'induced-hereditary'.

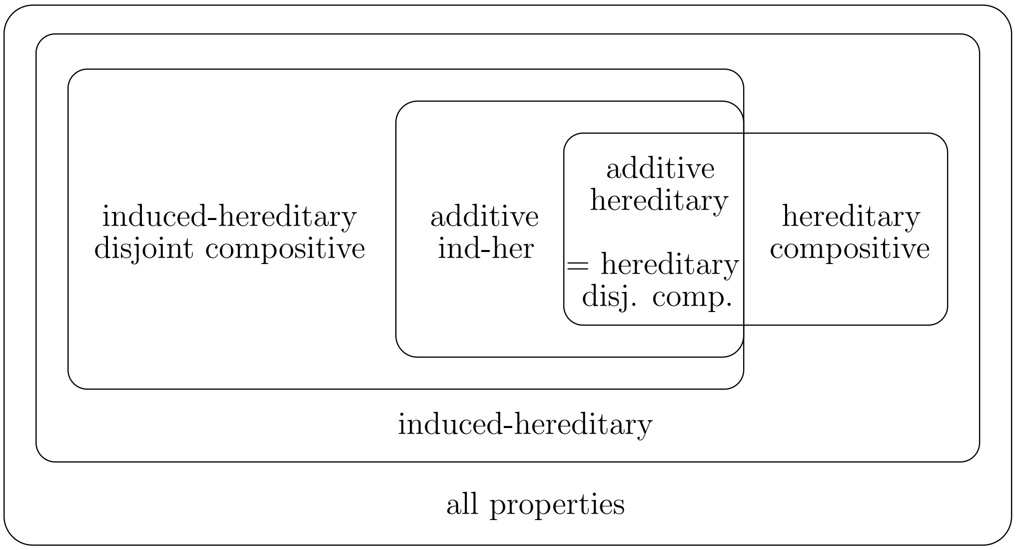


Figure 2.1: The additive and compositive classes that we consider.

- The four compositive classes we defined are each closed under multiplication, except for $\mathbb{L}^c_\leq$: $\{K_1\}$ is induced-hereditary compositive, but $\{K_1\}^2 = \{K_1, K_2, \overline{K_2}\}$ is not.

- Although every hereditary property is induced-hereditary, not all hereditary compositive properties are induced-hereditary compositive, and $\{K_1\}^2$ is again a suitable example. However, if an induced-hereditary compositive property is also hereditary, then it is clearly hereditary compositive.

- Every additive induced-hereditary property is induced-hereditary disjoint compositive, but not *vice versa*, as shown by the property of all complete graphs. However, a property is additive hereditary if and

	compositive	disjoint-compositive
hereditary	$\mathbb{L}^c$	$\mathbb{L}^{dc}$
induced-hereditary	$\mathbb{L}^c_\leq$	$\mathbb{L}^{dc}_\leq$

Table 2.1: Compositivity comes in four flavours.

only if it is hereditary disjoint compositive. All disjoint compositive properties must be infinite.

- The property that contains K_3 and all forests is clearly hereditary, but not hereditary compositive, as no graph in this class contains both K_3 and $K_{1,3}$. If we add in the disjoint union of K_3 with every forest, we get a property that *is* hereditary compositive, but not additive. Thus $\mathbb{L}^a \subset \mathbb{L}^c \subset \mathbb{L}$. Similarly, $\mathbb{L}^a_\leq \subset \mathbb{L}^{dc}_\leq \subset \mathbb{L}_\leq$, where the set of complete graphs, and any finite induced-hereditary property, show that the containments are strict.

We will discuss $\mathbb{L}^c_\leq$ in this section but not in subsequent chapters, because it is not closed under multiplication. Moreover, $\mathbb{L}^{dc} = \mathbb{L}^a$, so we are left with two new classes of properties that we will consider: hereditary compositive, and induced-hereditary disjoint compositive (for short, *indiscompositive*). Figure 2.1 shows the relationship between these classes and the classes of additive hereditary and induced-hereditary properties.

We first came to the notion of compositivity through the work of Scheinerman on intersection properties, and we spend the rest of this section giving his characterisations of the compositive classes, and expanding on them.

Let $\mathcal{F}$ be a family of sets; for any finite subfamily $\mathcal{F}' = \{S_1, \ldots, S_r\}$, where the S_i's are all distinct, the *intersection graph* $G(\mathcal{F}') = (V, E)$ is defined by taking $V = \mathcal{F}'$ and $E = \{(S_i, S_j) \mid S_i \cap S_j \neq \emptyset\}$ (e.g. Figure 2.2).

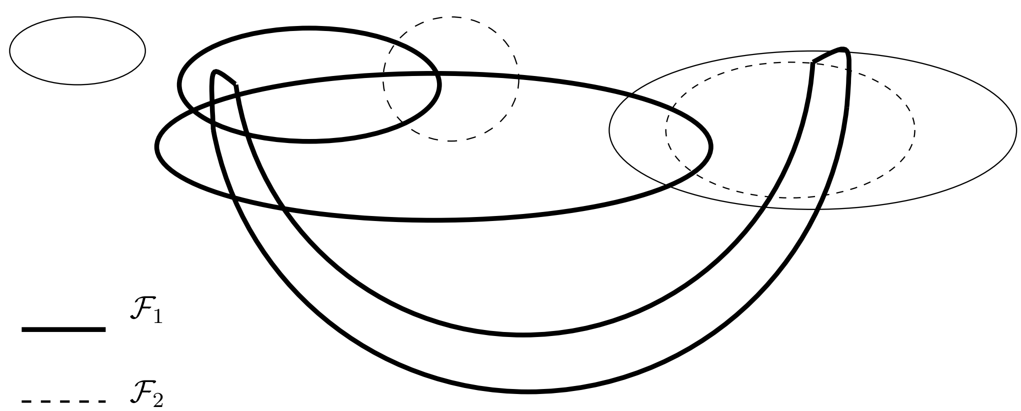


Figure 2.2: $G(\mathcal{F}_1 \cup \mathcal{F}_2)$ contains $G(\mathcal{F}_1)$ and $G(\mathcal{F}_2)$ as induced-subgraphs.

The *intersection property* $\mathcal{I}(\mathcal{F})$ consists of all such intersection graphs — $\mathcal{I}(\mathcal{F}) := \{G(\mathcal{F}') \mid \mathcal{F}' \subseteq \mathcal{F}, |\mathcal{F}'| \text{ finite}\}$. For example, if $\mathcal{F}$ is the set of intervals of the real line, $\mathcal{I}(\mathcal{F})$ is the class of interval graphs; more generally, if $\mathcal{F}$ is the set of d-dimensional rectangles, $\mathcal{I}(\mathcal{F})$ is the set of boxicity-d graphs. And if $\mathcal{F} = \{\{v_i, v_j\} \mid i, j \geq 0\}$, then $\mathcal{I}(\mathcal{F})$ is the set of line graphs. Scheinerman [55] showed that $\mathcal{P}$ is an intersection property if and only if $\mathcal{P}$ is induced-hereditary compositive, and gave many other equivalent definitions.

We will need some new concepts to state his result. A *generating set* for $\mathcal{P}$ is a set $\mathcal{G} \subseteq \mathcal{P}$ such that every graph in $\mathcal{P}$ is an induced-subgraph of some $G \in \mathcal{G}$; equivalently, $\mathcal{P}$ is the smallest induced-hereditary property that contains $\mathcal{G}$. A property has a generating set iff it is induced-hereditary. A generating set is *ordered* if it is a $\leq$-chain. A *generating graph* for $\mathcal{P}$ is a (finite or infinite) graph H such that $\mathcal{P} = \{G \mid G \leq H, |V(G)| \text{ finite}\}$.

For any $L \in \mathcal{P}$, $\mathcal{G}[L]$ is the set $\{G \in \mathcal{G} \mid L \leq G\}$. Mihók et al. introduced the set $\mathcal{G}[L]$, and stated that A $\Rightarrow$ B in Theorem 2.2.1; the rest of the theorem is due to Scheinerman. We include a complete proof here, both as background, and because it leads to a simplification (suggested by Jim Geelen) of the proof of the existence of a factorisation, specifically, of Theorems 3.2.2 and 4.3.4.

2.2.1. Theorem [51, 52, 55, 56]**.** *For any property $\mathcal{P}$, the following are equivalent:*

A. $\mathcal{P}$ is induced-hereditary compositive;

B. $\mathcal{P}$ has a generating set; moreover, for any graph $L \in \mathcal{P}$, and any generating set $\mathcal{G}$ of $\mathcal{P}$, $\mathcal{G}[L]$ is also a generating set;

C. $\mathcal{P}$ has a (finite or infinite) ordered generating set $H_1 \leq H_2 \leq \cdots$;

D. $\mathcal{P}$ has a generating graph H; and

E. $\mathcal{P}$ is an intersection property.

Proof: (A $\Rightarrow$ B). Since $\mathcal{P}$ is induced-hereditary, it is itself a generating set for $\mathcal{P}$. Now for any graph $G \in \mathcal{P}$, compositivity implies that there is some graph L'_G that contains both L and G; since $\mathcal{G}$ is a generating set there is some graph $L_G \in \mathcal{G}$ that contains L'_G, and thus $G \leq L_G \in \mathcal{G}[L]$.

(B $\Rightarrow$ C). Let $\mathcal{G}_1 = \{G_1, G_2, G_3, \ldots\}$ be a generating set for $\mathcal{P}$. Define $H_1 := G_1$; then $\mathcal{G}_2 := \mathcal{G}_1[H_1]$ is also a generating set, so it contains some graph

H_2 that contains G_2 (and also H_1). In general, for each $i > 1$, $\mathcal{G}_{i+1} := \mathcal{G}_i[H_i]$ contains some graph H_{i+1} containing H_i and G_{i+1}. Thus $H_1 \leq H_2 \leq H_3 \leq \cdots$ is an ordered generating set.

(C $\Rightarrow$ D). There is no loss of generality in assuming $|V(H_j)| = j$ for all j. If the ordered generating set is finite, say $H_1 \leq \cdots \leq H_r$, then take H to be H_r. Otherwise, for each j, we can label $V(H_j) = \{v_1, v_2, \ldots, v_j\}$ so that the vertices $\{v_1, \ldots, v_i\}$ induce H_i (as a labeled graph) whenever $i \leq j$. So if v_i and v_j are adjacent in some H_k, then they are adjacent in all subsequent H_r. We now let H be the infinite graph with vertices $\{v_1, v_2, \ldots\}$, where v_i and v_j are adjacent iff they are adjacent in some H_k. If $G \leq H$, with v_j being the vertex of $V(G)$ with largest index, then $G \leq H_j$ so that $G \in \mathcal{P}$; conversely, if G is in $\mathcal{P}$, then it is contained in some H_j, and therefore is contained in $H[v_1, \ldots, v_j]$.

(D $\Rightarrow$ E). Let $u \sim v$ denote that u and v are adjacent. Given the graph H with vertices $v_1, v_2, \ldots$, define the family of sets $\mathcal{F} := \{S_i, i \geq 1\}$, where $S_i := \{\{v_i, v_j\} \mid v_i \sim v_j\}$. Then $S_i \cap S_j \neq \emptyset$ iff $v_i \sim v_j$, so that $\mathcal{P} = \mathcal{I}(\mathcal{F})$.

(E $\Rightarrow$ A). If G_1 and G_2 are in $\mathcal{P} = \mathcal{I}(\mathcal{F})$, then $G_i = G(\mathcal{F}_i)$ for some finite $\mathcal{F}_i \subseteq \mathcal{F}$. If $G_i' \leq G_i$, then $V(G_i') = \mathcal{F}_i' \subseteq \mathcal{F}_i$, and $G_i' = G(\mathcal{F}_i')$ is in $\mathcal{I}(\mathcal{F}) = \mathcal{P}$, so $\mathcal{P}$ is induced-hereditary. Moreover, $G(\mathcal{F}_1 \cup \mathcal{F}_2)$ is a graph in $\mathcal{I}(\mathcal{F})$ that contains both $G(\mathcal{F}_1)$ and $G(\mathcal{F}_2)$ as induced-subgraphs, so $\mathcal{P}$ is induced-hereditary compositive. $\square$

A property need not have a unique generating graph. For example, if $\mathcal{P}$ is the set of all finite planar graphs, we can take the disjoint union of all planar graphs, or we can put paths between pairs of components of this infinite graph to get a connected generating graph; in fact, we can get infinitely many non-isomorphic generating graphs — at least one with exactly k components, for each $k \in \mathbb{N}$.

Bonato and Tardif [10] also give examples of what happen to be generating graphs for $\mathcal{P} := \{G \mid \text{every component of } G \text{ is a path}\}$. Let S be an infinite set of positive integers, and let P_S be the disjoint union of paths of length s, $s \in S$. There are uncountably many such graphs, and they are all induced-subgraphs of each other; in fact, Bonato and Tardif prove that for every infinite cardinal κ, there are 2^κ non-isomorphic graphs that are induced-subgraphs of each other.

Scheinerman proved the following result, and showed that α_1 could be arbitrarily large, or even infinite. We state the result for its own interest, but will not use it elsewhere.

2.2.2. Theorem [56]. *Let $\mathcal{P}$ be an induced-hereditary property. The following numbers are equal.*

(1) The minimum number α_1 such that $\mathcal{P}$ is the union of α_1 induced-hereditary compositive properties.

(2) The minimum number α_2 such that there are α_2 (finite or infinite) sequences $G_1^i \le G_2^i \le \cdots$ in $\mathcal{P}$, so that every graph in $\mathcal{P}$ is an induced-subgraph of some G_j^i.

(3) The maximum number α_3 of graphs in $\mathcal{P}$ that pairwise have no $\le$-composition in $\mathcal{P}$. $\square$

2.3 More characterisations

In this section we extend Theorem 2.2.1 to hereditary compositive and induced-hereditary disjoint compositive properties. With appropriate changes in the definitions, parts A through D of Theorem 2.2.1 still hold for hereditary compositive properties. Theorem 2.2.1(E) does not always hold in the hereditary case, since there are hereditary compositive properties (such as $\{K_1\}^2$) that are not induced-hereditary compositive.

In the hereditary context, $\mathcal{G}$ is a generating set for $\mathcal{P}$ if every graph in $\mathcal{P}$ is a subgraph of some $G \in \mathcal{G}$; it is ordered if its elements can be listed as $G_1 \subseteq G_2 \subseteq \cdots$. The set $\mathcal{G}[L]$ is $\{G \in \mathcal{G} \mid L \subseteq G\}$ and H is a generating graph for $\mathcal{P}$ if $\mathcal{P} = \{G \mid G \subseteq H, |V(G)| \text{ finite}\}$. If G is a *proper* induced-subgraph of H, we write $G < H$. A graph G is $\mathcal{P}$-*maximal* if $G \in \mathcal{P}$, but for all $e \notin G$, $G + e \notin \mathcal{P}$; the set of $\mathcal{P}$-maximal graphs is $\mathcal{M}(\mathcal{P})$ — graphs in $\mathcal{M}(\mathcal{P})$ and $\mathcal{F}_{\subseteq}(\mathcal{P})$ are both at the 'boundary' of $\mathcal{P}$, but on different sides (Figure 2.3).

2.3.1. Theorem. *For any property $\mathcal{P}$ the following are equivalent:*

A. $\mathcal{P}$ is hereditary compositive;

B. $\mathcal{P}$ has a generating set; moreover, for any graph $L \in \mathcal{P}$, and any generating set $\mathcal{G}$ of $\mathcal{P}$, $\mathcal{G}[L]$ is also a generating set;

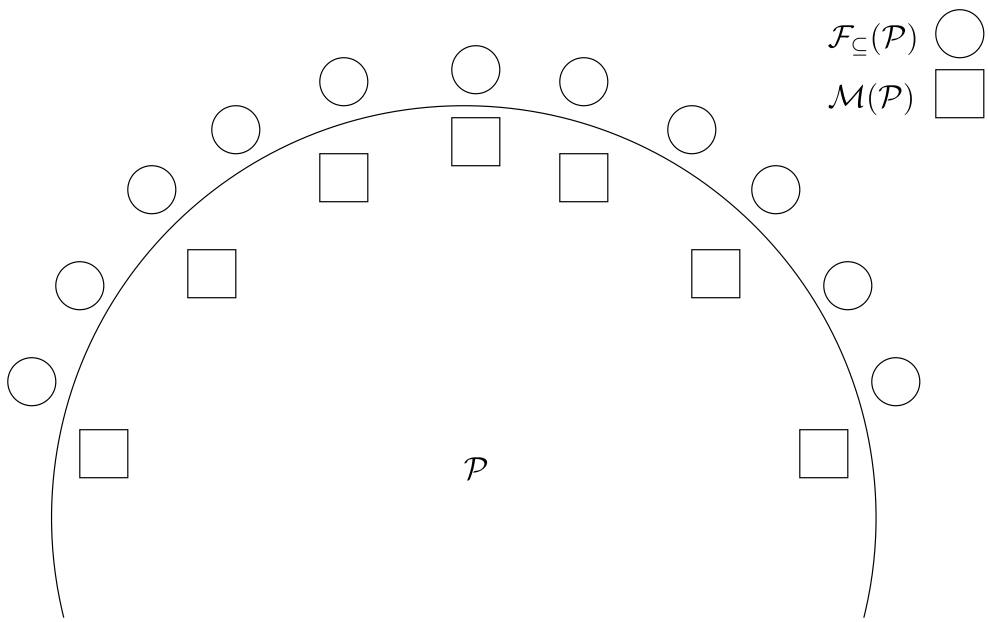


Figure 2.3: Maximal graphs and forbidden subgraphs at the boundary of $\mathcal{P}$.

C. $\mathcal{P}$ has a (finite or infinite) ordered generating set $G_1 \subseteq G_2 \subseteq \cdots$;

D. $\mathcal{P}$ has a (finite or infinite) ordered generating set $H_1 < H_2 < \cdots$ where each H_i is $\mathcal{P}$-maximal; and

E. $\mathcal{P}$ has a generating graph H.

Proof: D $\Rightarrow$ E $\Rightarrow$ A $\Rightarrow$ B $\Rightarrow$ C are the same as in Theorem 2.2.1.

(C $\Rightarrow$ D). We construct the H_i's as follows. Define $G_{j_1} := G_1$. For each $i \geq 1$, we add edges to G_{j_i} until we obtain some $\mathcal{P}$-maximal graph H_i. Then, let j_{i+1} be the least index such that $G_{j_{i+1}}$ contains H_i as a proper subgraph (or, if $H_i = G_{j_i}$ was the last graph in the original ordered generating set, we make it also the last graph of the new ordered generating set). The H_i's do in fact form an ordered generating set, because by construction $H_i \subseteq H_{i+1}$ for all i, and every G_k is eventually a subgraph of some H_i. Moreover, since H_i is $\mathcal{P}$-maximal, it must be an *induced*-subgraph of H_{i+1} (otherwise $H_i + e$ would be in $\mathcal{P}$, for some $e \notin H_i$). In fact, H_i is a proper induced subgraph of H_{i+1} by construction. $\square$

Disjoint compositive properties can be characterised by results analogous to Theorem 2.2.1. It is easier to consider first just the additive induced-hereditary properties; as before, there is a very similar result for additive hereditary properties, which are precisely the hereditary disjoint compositive properties. We use kG to denote the disjoint union of k copies of G. The last characterisation in the next result is folklore.

2.3.2. Theorem. *For any property $\mathcal{P}$ the following are equivalent:*

A. $\mathcal{P}$ is additive induced-hereditary;

B. $\mathcal{P}$ has a generating set; moreover, for any graph $L \in \mathcal{P}$, and any generating set $\mathcal{G}$ of $\mathcal{P}$, $\mathcal{G}[2L]$ is also a generating set;

C. $\mathcal{P}$ has an ordered generating set $H_1 \leq H_2 \leq \cdots$ such that $2H_i \leq H_{i+1}$ for all i;

D. $\mathcal{P}$ has a generating graph H such that $2H \leq H$; and

E. $\mathcal{F}_{\leq}(\mathcal{P})$ characterises $\mathcal{P}$, and contains only connected graphs.

Proof: (A $\Rightarrow$ B). Since $\mathcal{P}$ is induced-hereditary, it is itself a generating set for $\mathcal{P}$. For any graph $G \in \mathcal{P}$, $L_G := L \cup L \cup G$ is also in $\mathcal{P}$ by additivity. Since $\mathcal{G}$ is a generating set, L_G is contained in some $L'_G \in \mathcal{G}$; now $G \leq L_G \leq L'_G \in \mathcal{G}[2L]$, and G is arbitrary, so $\mathcal{G}[2L]$ generates $\mathcal{P}$.

(B $\Rightarrow$ C). Let $\mathcal{J}_1 = \{J_1, J_2, J_3, \ldots\}$ be a generating set for $\mathcal{P}$, and define $H_1 := J_1$. For each $i > 1$, take H_{i+1} to be a graph in $\mathcal{J}_{i+1} := \mathcal{J}_i[2H_i]$ that contains J_{i+1} (and $2H_i$).

(C $\Rightarrow$ A). Since $\mathcal{P}$ has a generating set, it is induced-hereditary. Now any G_1 and G_2 in $\mathcal{P}$ are contained in some H_{i_1} and H_{i_2}, respectively; without loss of generality, $H_{i_1} \leq H_{i_2}$, so $(G_1 \cup G_2) \leq (H_{i_2} \cup H_{i_2}) \leq H_{i_2+1}$, so $G_1 \cup G_2$ is in $\mathcal{P}$.

It is not immediately clear how to deduce D directly from C as we did previously — for example, if $\mathcal{P}$ is the set of graphs with each component a path, a natural ordered generating set would be $G_i := P_{3^i}$, from which we get the one-way infinite path as our graph H, which does not contain two copies of itself. We will therefore prove A $\Rightarrow$ D $\Rightarrow$ E $\Rightarrow$ A.

(A $\Rightarrow$ D). Take a generating set $\{G_1, G_2, \ldots\}$, and let H be the graph with infinitely many disjoint copies of each G_i. Clearly $2H \leq H$, and the property generated by H contains $\mathcal{P}$. In fact, H generates exactly $\mathcal{P}$, since any finite induced-subgraph of H is an induced-subgraph of $k_1G_{i_1}\cup\cdots\cup k_rG_{i_r}$ (for some k_j's and i_j's), and is thus in $\mathcal{P}$ by additivity and induced-heredity.

(D $\Rightarrow$ E). Since $\mathcal{P}$ has a generating graph, it is induced-heredita-ry, and is therefore characterised by $\mathcal{F}_{\leq}(\mathcal{P})$. Now suppose for contradiction that $F \in \mathcal{F}_{\leq}(\mathcal{P})$ is disconnected, say $F = F_1 \cup F_2$. By minimality, F_1 and F_2 are both in $\mathcal{P}$, so $F_1, F_2 \leq H$. But then $(F_1 \cup F_2) \leq (H \cup H) \leq H$, so F would be in $\mathcal{P}$.

(E $\Rightarrow$ A). $\mathcal{P}$ is induced-hereditary because it is characterised by forbidden induced-subgraphs. Suppose G_1 and G_2 are both in $\mathcal{P}$, but $G_1 \cup G_2$ is not. Then there is some $F \in \mathcal{F}_{\leq}(\mathcal{P})$, $F \leq G_1 \cup G_2$. Since F is connected, either $F \leq G_1$ or $F \leq G_2$, a contradiction. So $\mathcal{P}$ is additive. $\square$

It is now straightforward to generalise parts A—D of this result to induced-hereditary disjoint compositive properties, but generalising the proof is another matter. We will use two Ramsey results; the first is a standard result which can be used to prove the second one very neatly [39], but we give our own direct proof of the second part.

A *countably infinite biclique* is a complete bipartite graph with countably infinite partite sets V and W. For colours c and d, $B_{c,d}$ is an edge-coloured countably infinite biclique where we can label $V = \{v_1, v_2, \cdots\}$ and $W = \{w_1, w_2, \cdots\}$ so that v_iw_j is coloured c if $i < j$, and d if $i \geq j$ (see Figure 2.4). We note that $B_{c,d}$ contains a monochromatic infinite biclique iff $c = d$.

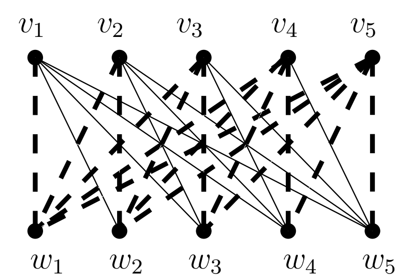


Figure 2.4: The first few vertices of $B_{c,d}$.

2.3.3. Theorem [39, Thms. 1.5 and 5.6]**.** *If the edges of an infinite clique are coloured with finitely many colours, then there is an infinite monochromatic clique. If the edges of a countably infinite biclique B are coloured with finitely many colours, then it contains a copy of $B_{c,d}$.*

Proof of the second part. Note that every infinite $V' \subseteq V$ and $W' \subseteq W$ define an infinite biclique contained in B. For each $v \in V'$, let $C(v, W')$ be the set of colours that appear on infinitely many of the edges $\{vw \mid w \in W'\}$; let $C(V', W')$ be the set of colours that appear in infinitely many of the sets $C(v, W')$, with $v \in V'$. Similarly, we define $C(w, V')$ and $C(W', V')$. Note that $C(V', W')$ and $C(W', V')$ are finite but non-empty.

Over all pairs (V', W'), where $V' \subseteq V$ and $W' \subseteq W$ are infinite sets, let V_0 and W_0 minimise the quantity $|C(V_0, W_0)| + |C(W_0, V_0)|$. If $V'' \subseteq V_0$ and $W'' \subseteq W_0$, then $C(V'', W'') \subseteq C(V_0, W_0)$ and $C(W'', V'') \subseteq C(W_0, V_0)$; so, if V'' and W'' are infinite, then

$$(*) \qquad C(V'', W'') = C(V_0, W_0) \text{ and } C(W'', V'') = C(W_0, V_0).$$

We fix colours $c \in C(V_0, W_0)$ and $d \in C(W_0, V_0)$, and construct $B_{c,d}$ in a straightforward manner. Let v_1 be a vertex in V_0 with $c \in C(v_1, W_0)$, and put v_1 in the set L of labeled vertices. Define $W_1 := \{w \in W_0 \mid v_1 w \text{ is coloured } c\}$; this is clearly an infinite set. By $(*)$, d is still in $C(W_1, V_0)$, so we can pick a vertex $w_1 \in W_1$ with $d \in C(w_1, V_0)$, put w_1 in L, and define $V_1 := \{v \in V_0 \setminus L \mid vw_1 \text{ is coloured } d\}$. By $(*)$, we have $c \in C(V_1, W_1)$; pick $v_2 \in V_1 \setminus L$ so that c is in $C(v_2, W_1)$; put v_2 in L, and define $W_2 := \{w \in W_1 \setminus L \mid v_2 w \text{ is coloured } c\}$.

By $(*)$, we can keep on labeling vertices indefinitely. At every stage, the labeled vertices induce a (finite) biclique in which $v_i w_j$ is coloured c if $i < j$, and d if $i \geq j$; we have therefore constructed $B_{c,d}$ in this manner. □

For a graph $L \in \mathcal{P}$, and a set $\mathcal{G} \subseteq \mathcal{P}$, $\mathcal{G}[2 \circledast L]$ is the set of graphs in $\mathcal{G}$ which contain two disjoint copies of L (possibly with some edges between them).

2.3.4. Theorem. *For a property $\mathcal{P}$ the following are equivalent:*

A. $\mathcal{P}$ is induced-hereditary disjoint compositive;

B. $\mathcal{P}$ has a generating set; moreover, for any graph $L \in \mathcal{P}$, and any generating set $\mathcal{G}$ of $\mathcal{P}$, $\mathcal{G}[2 \circledast L]$ is also a generating set;

C. $\mathcal{P}$ has an ordered generating set $G_1 \leq G_2 \leq \cdots$ such that every G_i contains two disjoint copies of G_{i-1}; and

D. $\mathcal{P}$ has a generating graph H that contains two disjoint copies of itself.

Proof: A $\Rightarrow$ B $\Rightarrow$ C $\Rightarrow$ A is proved as in Thm. 2.3.2.

(D $\Rightarrow$ A). Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$ be the vertex-sets of two disjoint copies of H contained in H. If G_1 and G_2 are in $\mathcal{P}$, then they are both induced-subgraphs of H, say $G_1 \cong H[x_1, \ldots, x_r]$, $G_2 \cong H[y_1, \ldots, y_s]$. Then $H[x_1, \ldots, x_r, y_1, \ldots, y_s]$ is a graph in $\mathcal{P}$ consisting of disjoint copies of G_1 and G_2.

(A $\Rightarrow$ D). This is also proved as in Thm. 2.3.2 — we take a generating set $\{G_1, G_2, \ldots\}$, and let H be the graph consisting of infinitely many disjoint copies of each G_i, but now we will need to put edges between these copies, and the point is how to choose those edges carefully (see Figure 2.5).

First of all, we put a "uniform" set of edges between the countably many copies of each G_i. Let i be fixed, and define $G_i^1 := G_i$ and $r_i := |V(G_i)|$. Label $V(G_i)$ with $v_1, \ldots, v_{r_i}$. For each $k > 1$, we can find a graph $G_i^k \in \mathcal{P}$ on kr_i vertices containing G_i^{k-1} and a k^{th} disjoint copy of G_i. We label $V(G_i^k)$ so that vertices $v_1^1, \ldots, v_{r_i}^{k-1}$ give us G_i^{k-1} (as a labeled graph). We label the remaining vertices $v_1^k, \ldots, v_{r_i}^k$ so that, ignoring the superscripts, we get the same labeled graph as G_i.

Now since r_i is finite, there are only finitely many (say s_i) configurations of edges that can be placed between two labeled copies of G_i. We colour the edges of an infinite tournament T with s_i colours so that, for $p \leq q$, the colour of arc (t_p, t_q) corresponds to the configuration between the p^{th} and q^{th} copies of G_i in G_i^q. By Theorem 2.3.3, T contains an infinite monochromatic tournament; in other words, we can form a graph H_i with countably many copies of G_i, each earlier copy being joined to each later one by the same configuration of edges, such that every finite subgraph of H_i is in $\mathcal{P}$.

We now look at what edges to put between H_1 and H_2. We will find a graph $H'_{1,2}$ that consists of disjoint copies of H_1 and H_2, such that every finite subgraph of $H'_{1,2}$ is in $\mathcal{P}$. We then construct an infinite biclique B with partition $\{v'_1, v'_2, \ldots\} \cup \{w'_1, w'_2, \ldots\}$, and colour $v'_i w'_j$ with a colour corresponding to the configuration of edges between the i^{th} copy of G_1 in H_1, and the j^{th} copy of G_2 in H_2.

Because G_1 and G_2 are finite, there will only be a finite number of colours, so by Theorem 2.3.3 our infinite biclique will contain a copy $H_{1,2}$ of $B_{c,d}$. If $V(H_{1,2})$ has partition $\{v_1, v_2, \ldots\} \cup \{w_1, w_2, \ldots\}$, then the subgraphs induced by $\{v_1, v_3, v_5, \ldots\} \cup \{w_1, w_3, w_5, \ldots\}$ and $\{v_2, v_4, v_6, \ldots\} \cup \{w_2, w_4, w_6, \ldots\}$ are both copies of $B_{c,d}$. Thus $H_{1,2} \leq H'_{1,2}$ contains two copies of itself.

In a similar manner, we then use $H_{1,2}$ and H_3 to construct a graph $H_{1,2,3}$ that contains two copies of itself; the only difference is that now the colour

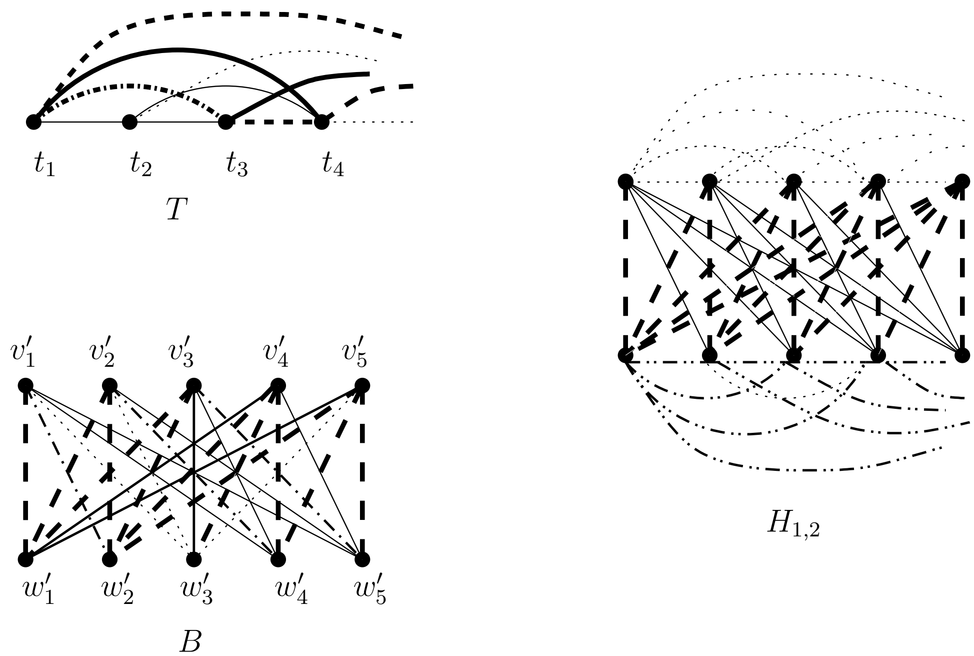


Figure 2.5: We take a monochromatic tournament from T to form H_1 or H_2, and a copy of $B_{c,d}$ from B, to form $H_{1,2}$.

of $v_i w_j$ represents the configuration of edges between the i^{th} copies of G_1 *and* G_2 in $H_{1,2}$, and the j^{th} copy of G_3 in H_3. Proceeding in this way, we can add H_4, H_5, and so on.

We still need to show how to obtain $H'_{1,2}$ from H_1 and H_2, so that every finite subgraph of $H'_{1,2}$ is in $\mathcal{P}$. For each k, let H_1^k be the graph consisting of the first k copies of G_1 in H_1; and similarly for H_2^k.

Let $X_1^k, \ldots, X_{z_k}^k$ be the graphs in $\mathcal{P}$ that can be formed by putting edges between disjoint copies of H_1^k and H_2^k; note that z_k is finite but positive. We form a graph with vertex set $V_1 \cup V_2 \cup \cdots$, where each V_k contains z_k vertices. The p^{th} vertex in V_k is joined to the q^{th} vertex of V_{k-1} if $X_p^k \geq X_q^{k-1}$; clearly, every vertex of V_k has at least one neighbour in V_{k-1}. By König's infinity lemma there is an infinite path, which corresponds to a sequence $X^1 \leq X^2 \leq \cdots$ where $X^k = X_{y_k}^k$ for some $y_k \leq z_k$. We now use the X^k's to construct $H_{1,2}$, in the same way as in Theorem 2.2.1(D). $\square$

2.4 Infinite graphs

Throughout this thesis we consider properties containing finite simple graphs; we digress briefly to discuss which of the characterisations given in this chapter still hold when we allow infinite graphs, or multigraphs with unbounded multiplicity. This will give an indication of the difficulties involved in considering infinite graphs.

The most obvious difficulty with infinite graphs is that we get infinite descending $\leq$-chains, and thus induced-hereditary properties are no longer characterised by *minimal* forbidden induced-subgraphs. Consider the property $\mathcal{P}$ of having finitely many edges, for example; no graph is at the 'boundary' of $\mathcal{P}$ — adding a vertex or any edge to a graph in $\mathcal{P}$ gives us another graph in $\mathcal{P}$, while for $G \notin \mathcal{P}$, we can always remove some vertex, or any edge, and remain outside $\mathcal{P}$. Yet, as pointed out to us by Jan Kratochvíl, $\mathcal{P}$ is characterised by three forbidden induced-subgraphs, namely the (countably) infinite clique, the infinite star and the infinite matching.

It is still true for infinite graphs that induced-hereditary properties are precisely the ones that have generating sets. Similarly, statements A and B of Theorem 2.2.1 are still equivalent, and in fact we have E $\Leftrightarrow$ D $\Rightarrow$ B $\Leftrightarrow$ A $\Leftarrow$ C (for E $\Rightarrow$ D, take $H = G(\mathcal{F})$).

Note that if we are considering properties of graphs with at most κ vertices, for some cardinal κ, then a generating graph for $\mathcal{P}$ is now a graph H such that $\mathcal{P} = \{G \leq H \mid |V(G)| \leq \kappa\}$, while the intersection property generated by a family $\mathcal{F}$ is $\mathcal{I}(\mathcal{F}) := \{G(\mathcal{F}') \mid \mathcal{F}' \subseteq \mathcal{F}, |\mathcal{F}'| \leq \kappa\}$. Besides, an ordered generating set now need not be countable.

To see that B $\not\Rightarrow$ D, and C $\not\Rightarrow$ D, consider the smallest additive induced-hereditary property $\mathcal{P}_\infty$ that contains the two-way countably infinite path $\overleftrightarrow{P}_\infty$. Let $\overrightarrow{P}_\infty$ be the one-way countably infinite path, let $\mathcal{L}$ be the property containing disjoint unions of at most countably many finite paths, and note that we must have $\kappa \geq \aleph_0$, where $\aleph_0 = |\mathbb{N}|$. We have

$$\mathcal{P}_\infty = \{k_1 \overleftrightarrow{P}_\infty \cup k_2 \overrightarrow{P}_\infty \cup L \mid 0 \leq k_1, k_2 < \aleph_0, L \in \mathcal{L}\}.$$

This property has an ordered generating set $\overleftrightarrow{P}_\infty \leq 2\overleftrightarrow{P}_\infty \leq \cdots$. However, a generating graph for $\mathcal{P}_\infty$ must contain $\aleph_0 \overleftrightarrow{P}_\infty$, and this must then be a graph in $\mathcal{P}$, a contradiction.

It is not clear whether B $\overset{?}{\Rightarrow}$ C or D $\overset{?}{\Rightarrow}$ C. One idea for proving the latter implication is to impose an arbitrary order $\prec$ on $V(H)$, and for each

$v \in V(H)$ define $S_v := \{u \in V(H) \mid u \preceq v\}$ and $H_v := H[S_v]$. The problem is that we might have $|S_v| > \kappa$.

As above, in Theorem 2.3.1 we have D $\Rightarrow$ C $\Rightarrow$ B $\Leftrightarrow$ A $\Leftarrow$ E, but B $\overset{?}{\Rightarrow}$ C is still unclear. We also have (B,C) $\not\Rightarrow$ (D,E): if each graph in $\mathcal{Q}$ is the union of a finite star and a (finite or infinite) number of isolated vertices, the only $\mathcal{Q}$-maximal graphs are the finite stars, which do not generate $\mathcal{Q}$. This example also fails to have a generating graph, because such a graph must contain an infinite star, which would then be in $\mathcal{Q}$. It is also unclear whether E $\overset{?}{\Rightarrow}$ C, or E $\overset{?}{\Rightarrow}$ D.

In Theorem 2.3.2 (and, similarly, in Theorem 2.3.4), we have (C,D,E) $\Rightarrow$ B $\Leftrightarrow$ A. However, (B,C) $\not\Rightarrow$ (D,E) — the property $\mathcal{P}$ of having finitely many edges and at most countably many vertices has no generating graph, and no *minimal* forbidden subgraphs; but if $\{G_1, G_2, \ldots\}$ is the (countable) set of all finite connected graphs, and G_0 is the countable graph with no edges, then $(G_0 \cup G_1) \leq (G_0 \cup 2G_1 \cup G_2) \leq (G_0 \cup 4G_1 \cup 2G_2 \cup G_3) \leq \cdots$ is a generating set for $\mathcal{P}$ as required in Theorem 2.3.2(C). As before, B $\overset{?}{\Rightarrow}$ C $\overset{?}{\Leftarrow}$ D $\overset{?}{\Rightarrow}$ E $\overset{?}{\Rightarrow}$ (C,D) are still open.

If we consider finite multigraphs, it makes sense to define G to be an induced-subgraph of H if, whenever two vertices are joined by k edges in H, they are joined by k' edges in G, for some $1 \leq k' \leq k$. A generating graph is now a graph H such that $\mathcal{P}$ is the set of all induced-subgraphs of H with finitely many vertices and edges. However, Theorem 2.3.1(C) $\not\Rightarrow$ D — $\mathcal{P} = \{k(u,v) \mid 0 \leq k < \infty\}$ is an ordered generating set for itself, but there is no $\mathcal{P}$-maximal graph.

2.5 Compositivity in terms of forbidden subgraphs

We now look briefly at the question of characterising compositive or disjoint compositive properties in terms of $\mathcal{F}_{\subseteq}(\mathcal{P})$ or $\mathcal{F}_{\leq}(P)$. This seems to be quite difficult in general, but we have a characterisation for properties with exactly one minimal forbidden (induced-)subgraph. For a (finite or infinite) set of graphs $\{G_1, G_2, \ldots\}$, we define $\mathrm{Forb}_{\subseteq}(G_1, G_2, \ldots) := \{G \mid \forall i,\ G_i \nsubseteq G\}$, and $\mathrm{Forb}_{\leq}(G_1, G_2, \ldots) := \{G \mid \forall i,\ G_i \nleq G\}$. In particular, if S is an antichain under $\subseteq$ or $\leq$, then $\mathcal{F}_{\subseteq}(\mathrm{Forb}_{\subseteq}(S)) = S$ and $\mathcal{F}_{\leq}(\mathrm{Forb}_{\leq}(S)) = S$, respectively.

Note that for any graph $G \neq K_1$, the property $\mathrm{Forb}_{\leq}(G)$ is induced-hereditary disjoint compositive, so any characterisation cannot consider the minimal forbidden induced-subgraphs individually, as in Theorem 2.3.2(E). This is not true for minimal forbidden subgraphs.

2.5.1. Proposition. *Let $\mathcal{P}$ be hereditary with exactly one minimal forbidden subgraph, say $\mathcal{P} = \mathrm{Forb}_{\subseteq}(G)$. Then exactly one of the following is true:*

A. *G is connected and $\mathcal{P}$ is additive;*

B. *$G = \overline{K}_r$, and $\mathcal{P} = \{H \mid |V(H)| < r\}$, so $\mathcal{P}$ is compositive but not additive; or*

C. *$\mathcal{P}$ is not even compositive.*

Proof: It is easy to check that if G is connected, then $\mathcal{P}$ is additive, and if $G = \overline{K}_r$ then $\mathcal{P}$ contains exactly the graphs with fewer than r vertices. So now let G have components $C_1, \ldots, C_r, r \geq 2$, not all trivial; we want to show that $\mathrm{Forb}_{\subseteq}(G)$ is not compositive. If the C_i's are not all isomorphic, then let C_1 have the fewest edges (so $C_j \nsubseteq C_1$ unless $C_1 \cong C_j$); $X := G - C_1$ is in $\mathcal{P}$, and so is the graph Y consisting of $|V(G)|$ disjoint copies of C_1. But in any graph H containing X and Y as subgraphs, the vertices of X cannot intersect all the copies of C_1 in Y, so H contains a copy of G as a subgraph, and thus $H \notin \mathcal{P}$.

If the C_i's are all isomorphic to C, then $|V(C)| \geq 2$ since we are not in Case B. Let v be an arbitrary vertex of C, and let Y contain $|V(G)|$ copies of C (call these *pendant copies* of C) with the $|V(G)|$ copies of v all identified. Note that in $Y - v$ no component can contain C, and thus in Y any copy of C (whether pendant or not) uses v; in particular Y does not contain two vertex-disjoint copies of C, so Y is in $\mathcal{P}$. The complete graph X on $|V(G)| - 1$ vertices is also in $\mathcal{P}$. Let H contain both X and Y as subgraphs. Then there must be some pendant copy of C in Y that does not intersect X except, possibly, in v. This pendant copy, along with $X - v$, contains r disjoint copies of C as a subgraph; thus H is not in $\mathcal{P}$. $\square$

The *join* $G_1 + G_2$ of two graphs consists of disjoint copies of G_1 and G_2 with all possible edges between them.

2.5.2. Proposition [56]**.** *Let $\mathcal{P}$ be induced-hereditary. If $\mathcal{P}$ is not induced-hereditary disjoint compositive, then at least one graph in $\mathcal{F}_{\leq}(\mathcal{P})$ is disconnected, and at least one graph in $\mathcal{F}_{\leq}(\mathcal{P})$ has disconnected complement.*

Proof: If the minimal forbidden induced-subgraphs are all connected, then $\mathcal{P}$ is additive (Theorem 2.3.2(E)), that is, it is closed under taking disjoint unions. Similarly, if the minimal forbidden induced-subgraphs all have connected complements, then none of them is a join, and $\mathcal{P}$ is closed under taking joins. □

2.5.3. Corollary. *Let $\mathcal{P}$ be induced-hereditary. If $\mathcal{P} = \mathrm{Forb}_{\leq}(G)$, then $\mathcal{P}$ is disjoint compositive; furthermore, $\mathcal{P}$ is additive if and only if G is connected. If $\mathcal{P} = \mathrm{Forb}_{\leq}(G, H)$, then $\mathcal{P}$ must be disjoint compositive unless exactly one of G and H is connected.* □

A logical step might be to look at properties of the form $\mathcal{P} = \mathrm{Forb}_{\leq}(G, H)$, where G is a join, and H is disconnected, and characterise those cases where $\mathcal{P}$ is compositive or disjoint compositive. However, it seems that this is quite a bit more complicated than classifying properties of the form $\mathrm{Forb}_{\subseteq}(G)$ as in Proposition 2.5.1.

Chapter 3

Unique factorisation — hereditary compositive properties

In this chapter we prove that hereditary compositive properties are uniquely factorisable into irreducible hereditary compositive properties. We also show that additive hereditary properties are uniquely factorisable into irreducible additive hereditary properties.

We start off in Section 3.1 with some basic definitions and results from Mihók et al. [51]. In Section 3.2 we reproduce their canonical factorisation and, after explaining the shortcoming in their uniqueness proof, we give our own proof in Section 3.3.

The chronological development, as usual, was quite different from the final presentation. Kratochvíl and Mihók proved unique factorisation for a significant class of additive hereditary properties in [47], the proof depending on the structure of those properties (and in the spirit of the proof we give here). Mihók et al. [51] then gave their canonical factorisation for all additive hereditary properties, and Mihók generalised this in [52] to additive induced-hereditary properties. We established uniqueness first for additive hereditary properties, and exactly the same proof turned out to work for hereditary compositive properties. We then generalised the proof to additive induced-hereditary properties and, later, to induced-hereditary disjoint compositive properties.

After showing these results to Mihók, he came up with a somewhat simpler proof of Theorems 3.3.1 and 4.3.3, which appears in [6]. In that paper, the results are stated for directed edge-coloured hypergraphs, as the proofs carry over with only minor changes to reflect the use of hyperedges, as an interested reader may check.

In the next chapter we prove unique factorisation for additive induced-hereditary properties, and for indiscompositive properties. Although the proofs of unique factorisation for $\mathbb{L}^c$ and $\mathbb{L}^a$ are the same, only the latter can be deduced from the result for $\mathbb{L}^a_{\leq}$ or $\mathbb{L}^{dc}_{\leq}$ (cf. Proposition 5.1.1). It seems that $\mathbb{L}^c$ requires a proof that is independent from the next chapter. In any case, the familiarity gained in this chapter will be helpful in tackling the more difficult proofs of the next one, although Chapter 4 can also be read independently.

Finally, we note that the structure of the canonical factorisation in Theorem 3.2.2 can be used to prove part of the uniqueness result (Theorem 3.3.2). Our proofs of Theorems 3.3.1 and 3.3.2, however, make no use of the structure of the canonical factors; they rely only on the more elementary aspects of [51].

3.1 Hereditary properties — the groundwork

The preliminary definitions and results in this section are adapted from [51]. For convenience, we repeat some of the definitions introduced in various parts of Chapter 2. The smallest hereditary property that contains a set $\mathcal{G}$ is denoted by $\mathcal{G}_{\subseteq}$. This is the hereditary property *generated by* $\mathcal{G}$, or that $\mathcal{G}$ *generates*. $\mathcal{G}$ is a *generating set* for $\mathcal{P}$ if $\mathcal{G}_{\subseteq} = \mathcal{P}$. It is easily seen that

$$\mathcal{G}_{\subseteq} \;=\; \{G \mid \exists\, H \in \mathcal{G},\ G \subseteq H\}.$$

The *completeness* $c(\mathcal{P})$ of a hereditary property $\mathcal{P}$ is $\max\{k : K_k \in \mathcal{P}\}$, where K_k is the complete graph on k vertices; clearly, $c(\mathcal{Q} \circ \mathcal{R}) = c(\mathcal{Q}) + c(\mathcal{R})$. Thus, any factorisation of a hereditary property $\mathcal{P}$ has at most $c(\mathcal{P})$ factors. We note that $c(\mathcal{P}) + 1 = \min\{|V(H)| \mid H \notin \mathcal{P}\} = \min\{|V(H)| \mid H \in \mathcal{F}_{\subseteq}(\mathcal{P})\}$. In some of the literature, the convention is to define $c(\mathcal{P}) := \max\{k : K_{k+1} \in \mathcal{P}\}$.

The *join* $G_1 + \cdots + G_n$ of n graphs $G_1, \ldots, G_n$ consists of disjoint copies of the G_i's, and all edges between $V(G_i)$ and $V(G_j)$, for $i \neq j$. A graph G is *decomposable* if it is the join of two graphs; otherwise, G is *indecomposable*. It is easy to see that G is decomposable if and only if its complement $\overline{G}$ is disconnected; G is the join of the complements of the components of $\overline{G}$, so every decomposable graph can be expressed uniquely as the join of indecomposable subgraphs, the *ind-parts* of G. The number of ind-parts of G is the *decomposability number* $dc(G)$ of G.

For a hereditary property $\mathcal{P}$, a graph G is $\mathcal{P}$*-strict* if $G \in \mathcal{P}$ but $G + K_1 \notin \mathcal{P}$. The set $\mathcal{M}(\mathcal{P})$ of $\mathcal{P}$-maximal graphs is defined as:

$$\begin{aligned} \mathcal{M}(n, \mathcal{P}) &:= \{G \in \mathcal{P} \mid |V(G)| = n \text{ and for all } e \notin E(G), G + e \notin \mathcal{P}\}; \\ \mathcal{M}(\mathcal{P}) &:= \bigcup_{n=1}^{\infty} \mathcal{M}(n, \mathcal{P}) \end{aligned}$$

Note that, for $1 \leq n \leq c(\mathcal{P})$, $M(n, \mathcal{P}) = \{K_n\}$. Since graphs with fewer than $c(\mathcal{P})$ vertices are not $\mathcal{P}$-strict, it is useful to define

$$\mathcal{M}^*(\mathcal{P}) := \bigcup_{n=c(\mathcal{P})}^{\infty} \mathcal{M}(n, \mathcal{P}).$$

This is the set of $\mathcal{P}$-strict $\mathcal{P}$-maximal graphs.

3.1.1. Lemma [19, 51]**.** *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$, where the $\mathcal{P}_i$'s are hereditary graph properties. A graph G belongs to $\mathcal{M}(\mathcal{P})$ if and only if, for every $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-colouring $(V_1, \ldots, V_m)$ of G, the following holds: $G[V_i] \in \mathcal{M}(\mathcal{P}_i)$ for $i = 1, \ldots, m$, and $G = G[V_1] + \cdots + G[V_m]$. Moreover, if $G \in \mathcal{M}^*(\mathcal{P})$, then it is $\mathcal{P}$-strict, each $G[V_i]$ is $\mathcal{P}_i$-strict, and is in $\mathcal{M}^*(\mathcal{P}_i)$; in particular, each V_i is non-empty.*

Proof: Let G be in $\mathcal{M}(\mathcal{P})$, and let $(V_1, \ldots, V_m)$ be a $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-colouring of G. If, for some i, $G[V_i]$ is not in $\mathcal{M}(\mathcal{P}_i)$, then there are $u, v \in V_i$ such that $uv \notin G[V_i]$ and $G[V_i] + uv \in \mathcal{P}_i$, so $G + uv \in \mathcal{P}$. If $G \neq G[V_1] + \cdots + G[V_m]$, then there are vertices $u \in V_i, v \in V_j$, for some $i \neq j$, such that $uv \notin G$; but then $G + uv$ is in $\mathcal{P}$.

Conversely, suppose G is not in $\mathcal{M}(\mathcal{P})$. Then there is some edge $uv \notin G$, such that $G + uv \in \mathcal{P}$. Let $(V_1, \ldots, V_m)$ be a $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-colouring of $G + uv$; because the $\mathcal{P}_i$'s are hereditary, it is also a $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-colouring of G. If u and v are in the same V_i, then $G[V_i]$ is not $\mathcal{P}_i$-maximal; otherwise, $G \neq G[V_1] + \cdots + G[V_m]$.

Suppose now that G is in $\mathcal{M}^*(\mathcal{P})$. If $|V(G)| = c(\mathcal{P})$, then $G = K_{c(\mathcal{P})}$ and $G + K_1 = K_{c(\mathcal{P})+1}$ is not in $\mathcal{P}$. Otherwise, G is not complete, so there are non-adjacent vertices u and v; then $G+K_1$ contains a subgraph isomorphic to $G+uv$. Since G is $\mathcal{P}$-maximal, $G+uv$ is not in $\mathcal{P}$, and, since $\mathcal{P}$ is hereditary, $G+K_1$ is not in $\mathcal{P}$. In either case, G is $\mathcal{P}$-strict.

Suppose, without loss of generality, that $G[V_1]$ were not $\mathcal{P}$-strict. Then $G+K_1 = (G[V_1]+K_1)+G[V_2]+\cdots+G[V_m]$, which is in $\mathcal{P}$ since $G[V_1]+K_1$ is in $\mathcal{P}_1$, a contradiction. So each $G[V_i]$ is $\mathcal{P}_i$-strict; we already know it is in $\mathcal{M}(\mathcal{P}_i)$, and the graphs in $\mathcal{M}(\mathcal{P}_i)\setminus\mathcal{M}^*(\mathcal{P}_i)$ are not $\mathcal{P}_i$-strict, proving the last part. □

It follows that, if $\mathcal{P}$ is reducible, then every graph in $\mathcal{M}^*(\mathcal{P})$ is decomposable. We note that the join of a $\mathcal{Q}$-maximal graph G and an $\mathcal{R}$-maximal graph H need not be $(\mathcal{Q}\circ\mathcal{R})$-maximal; for example, take G to be complete, $|V(G)| < c(\mathcal{Q})$, and H not complete.

3.1.2. Lemma [51]**.** *Let $\mathcal{P}$ be a hereditary property and let $G \in \mathcal{M}^*(\mathcal{P})$, $H \in \mathcal{P}$. If $G \subseteq H$, then $dc(H) \le dc(G)$. If we have equality, with $G = G_1 + \cdots + G_n$ and $H = H_1 + \cdots + H_n$ being the respective expressions as joins of ind-parts, then we can relabel the ind-parts of H so that each G_i is an induced-subgraph of H_i.*

Proof: Each component $\overline{G_i}$ of $\overline{G}$ is contained in a component $\overline{H_j}$ of $\overline{H}$. If, for some k, $\overline{H_k}$ were disjoint from $\overline{G}$, then $G+H_k \subseteq H$, whence $G+K_1 \in \mathcal{P}$, a contradiction. □

Clearly $\mathcal{M}^*(\mathcal{P})_{\subseteq} = \mathcal{P}$, but if $\mathcal{P}$ is additive this will not be the unique generating set. The *decomposability number* of a set $\mathcal{G}$ of graphs is $dc(\mathcal{G}) := \min\{dc(G) \mid G \in \mathcal{G}\}$; the *decomposability number* of a hereditary property $\mathcal{P}$ is $dc(\mathcal{P}) := dc(\mathcal{M}^*(\mathcal{P}))$.

3.1.3. Lemma [51]**.** *If $\mathcal{G}$ generates the hereditary property $\mathcal{P}$, then $dc(\mathcal{G}) \le dc(\mathcal{M}^*(\mathcal{P}))$, with equality if $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$.*

Proof: Let G be a graph in $\mathcal{M}^*(\mathcal{P})$ with the minimum decomposability $dc(\mathcal{M}^*(\mathcal{P}))$. Since $\mathcal{G}$ generates $\mathcal{P}$, there is some $H \in \mathcal{G}$ that contains G. By Lemma 3.1.2, $dc(H) \le dc(G)$, so $dc(\mathcal{G}) \le dc(H) \le dc(G) = dc(\mathcal{M}^*(\mathcal{P})$. If $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$, then trivially $dc(\mathcal{G}) \ge dc(\mathcal{M}^*(\mathcal{P}))$, so we have equality. □

For $\mathcal{G} \subseteq \mathcal{P}$ and $H \in \mathcal{P}$, let $\mathcal{G}[H] := \{G \in \mathcal{G} \mid H \subseteq G\}$. Note that, if $\mathcal{G}$ is an ordered generating set, say $G_1 \subseteq G_2 \subseteq \cdots$, and G_k is the first graph to contain G, then $\mathcal{G}[G]$ is simply $G_k \subseteq G_{k+1} \subseteq \cdots$. The following lemma was already stated as Theorem 2.3.1(B).

3.1.4. Lemma [51]**.** *Let $\mathcal{G}$ generate the hereditary compositive property $\mathcal{P}$, and let H be an arbitrary graph in $\mathcal{P}$. Then $\mathcal{G}[H]$ also generates $\mathcal{P}$.* □

For a generating set $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$, let $\mathcal{G}^{\downarrow} := \{G \in \mathcal{G} \mid dc(G) = dc(\mathcal{P})\}$.

3.1.5. Lemma [51]**.** *If $\mathcal{G} \subseteq \mathcal{M}^*(\mathcal{P})$ generates the hereditary compositive property $\mathcal{P}$, then so does $\mathcal{G}^{\downarrow}$.*

Proof: By Lemma 3.1.3, $dc(\mathcal{G}) = dc(\mathcal{M}^*(\mathcal{P}))$, so there is a graph $H \in \mathcal{G}$ with $dc(H) = dc(\mathcal{P})$. Now by Lemma 3.1.4, $\mathcal{G}[H]$ generates $\mathcal{P}$, and by Lemma 3.1.2, $\mathcal{G}[H] \subseteq \mathcal{G}^{\downarrow}$. □

3.2 The canonical factorisation

In this section we reproduce Mihók et al.'s proof [51] that every hereditary compositive property $\mathcal{P}$ has a factorisation into $dc(\mathcal{P})$ hereditary compositive properties. Mihók et al. had stated that their factorisation is unique; we reproduce their proof in this section, and explain why it does not establish their claim. Recall from Sections 1.2 and 2.2 that:

- the classes of hereditary, hereditary compositive and additive hereditary properties are, respectively, $\mathbb{L}$, $\mathbb{L}^c$ and $\mathbb{L}^a$. The class of all properties is $\mathbb{U}$.

- if $\mathbb{P}$ is a class of properties, then $\mathcal{P} \in \mathbb{P}$ is *reducible over* $\mathbb{P}$ if it is the product of at least two properties from $\mathbb{P}$; otherwise, it is *irreducible over* $\mathbb{P}$. When $\mathbb{P} = \mathbb{U}$, we just say that $\mathcal{P}$ is *irreducible*.

Recall from Proposition 2.1.3 that a hereditary property is irreducible iff it is irreducible over $\mathbb{L}$; we can therefore talk unambiguously about irreducible hereditary properties. A hereditary property $\mathcal{P}$ is *indecomposable* if $dc(\mathcal{P}) = 1$. By Lemma 3.1.1 such a property must be irreducible, although [51, Example 4.2] there are also irreducible hereditary properties that are decomposable.

3.2.1. Lemma. *For hereditary properties $\mathcal{Q}$ and $\mathcal{R}$, $dc(\mathcal{Q} \circ \mathcal{R}) \geq dc(\mathcal{Q}) + dc(\mathcal{R})$. Thus, if $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$, then $m \leq dc(\mathcal{P})$; if we have equality, then each $\mathcal{P}_i$ is indecomposable.* $\square$

The set $\mathcal{M}^*(\mathcal{P})$ clearly generates the hereditary compositive property $\mathcal{P}$; by Lemmas 3.1.5 and 3.1.1, there is a generating set $\mathcal{G}^* \subseteq \mathcal{M}^*(\mathcal{P})$ whose graphs are $\mathcal{P}$-strict, with decomposability $dc(\mathcal{P})$. For a graph G the set of its ind-parts is denoted by $Ip(G)$. The set of all ind-parts from $\mathcal{G}^*$ is $I_{\mathcal{G}} := \cup(Ip(G) \mid G \in \mathcal{G}^*)$.

For $F \in I_{\mathcal{G}}$ and $G \in \mathcal{G}^*$, $m(F,G)$ is the *multiplicity of F in G*: the number of different (possibly isomorphic) ind-parts of G which contain F as a subgraph. The *multiplicity of F in $\mathcal{G}^*$* is $m(F) = \max\{m(F,G) \mid G \in \mathcal{G}^*\}$; by choice of $\mathcal{G}^*$, $1 \leq m(F) \leq dc(\mathcal{P})$.

We note that there is an alternative proof of Theorem 3.2.2 due to Jim Geelen; we present its induced-hereditary analog in Theorem 4.3.4.

3.2.2. Theorem [51]**.** *A hereditary compositive property $\mathcal{P}$ has a factorisation into $dc(\mathcal{P})$ (necessarily indecomposable) hereditary compositive factors. Moreover, when $\mathcal{P}$ is additive, the factors can be taken to be additive too.*

Proof: We proceed by induction on $dc(\mathcal{P})$. If $dc(\mathcal{P}) = 1$ there is nothing to do. So let every graph $G \in \mathcal{M}^*(\mathcal{P})$ be decomposable. We will factor either into $n := dc(\mathcal{P})$ properties, or into properties $\mathcal{Q}, \mathcal{R}$ such that $dc(\mathcal{P}) = dc(\mathcal{Q}) + dc(\mathcal{R})$.

Case 1. For some $F \in I_{\mathcal{G}}$, $m(F) < dc(\mathcal{P})$.

Let k be $m(F)$, and let $G \in \mathcal{G}^*$ be a graph for which $m(F,G) = k$. By Lemma 3.1.4, $\mathcal{G}^*[G]$ generates $\mathcal{P}$; by Lemma 3.1.2, for every generator $H \in \mathcal{G}^*[G]$, $m(F,H) = k$, so $\mathcal{G}^*_F := \{G' \in \mathcal{G}^* \mid m(F,G') = k\}$ is a generating set. For $H \in \mathcal{G}^*_F$, let H_F be the subgraph induced by (the vertices of) the k ind-parts which contain F, and $H_{\overline{F}}$ the subgraph induced by (the vertices of) the $n-k$ other ind-parts. Let the hereditary properties $\mathcal{Q}_F$ and $\mathcal{Q}_{\overline{F}}$ be generated by $\{H_F \mid H \in \mathcal{G}^*_F\}$ and $\{H_{\overline{F}} \mid H \in \mathcal{G}^*_F\}$, respectively.

We claim that $\mathcal{P} = \mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$. It is easy to see that $\mathcal{P} \subseteq \mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$. Conversely, let H be in $\mathcal{Q}_F \circ \mathcal{Q}_{\overline{F}}$. Then $H \subseteq H^1_F + H^2_{\overline{F}}$, for some $H^1, H^2 \in \mathcal{G}^*_F$. Let H' be a graph in $\mathcal{G}^*$ that contains both H^1 and H^2 as subgraphs. By Lemma 3.1.2, and because the maximum multiplicity of F in $\mathcal{G}^*$ is k, $H^1_F \subseteq H'_F$ and $H^2_{\overline{F}} \subseteq H'_{\overline{F}}$. Thus $H \subseteq H^1_F + H^2_{\overline{F}} \subseteq H'_F + H'_{\overline{F}} = H' \in \mathcal{P}$, implying $H \in \mathcal{P}$.

To establish compositivity of $\mathcal{Q}_F$, consider $G_F, H_F \in \mathcal{Q}_F$, for some $G, H \in \mathcal{G}_F^*$. Because $\mathcal{G}_F^*$ generates $\mathcal{P}$, there is some $L \in \mathcal{G}_F^*$ that contains both G and H as subgraphs. By Lemma 3.1.2, $G_F \subseteq L_F$ and $H_F \subseteq L_F$, so $L_F \in \mathcal{Q}_F$ contains both G_F and H_F as subgraphs.

If $\mathcal{P}$ is additive, then we can find $L' \in \mathcal{G}_F^*$ that contains $G \cup H$ as a subgraph. Since $G_F \subseteq L'_F$ and $H_F \subseteq L'_F$, $(G_F \cup H_F) \subseteq L'_F \in \mathcal{Q}_F$, and thus $(G_F \cup H_F) \in \mathcal{Q}_F$. Compositivity or additivity of $\mathcal{Q}_{\overline{F}}$ is proved similarly.

Finally, $\mathcal{Q}_F$ and $\mathcal{Q}_{\overline{F}}$ are defined by generating sets whose members have decomposability k and $n-k$, respectively. Using Lemmas 3.1.1 and 3.1.3, we have $n = dc(\mathcal{P}) \geq dc(\mathcal{Q}_F) + dc(\mathcal{Q}_{\overline{F}}) \geq k + (n-k) = n$, therefore $dc(\mathcal{Q}_F) = k$ and $dc(\mathcal{Q}_{\overline{F}}) = n-k$. Since $k < n$ and $n-k < n$, it follows that $\mathcal{P}$ has a factorisation into $dc(\mathcal{P})$ factors.

Case 2. For each $F \in I_{\mathcal{G}}$, $m(F) = n := dc(\mathcal{P})$.

Let $\mathcal{Q}$ be the induced-hereditary property generated by $I_{\mathcal{G}}$. It is easy to see that $\mathcal{P} \subseteq \mathcal{Q}^n$. The reverse inclusion, $\mathcal{Q}^n \subseteq \mathcal{P}$, and the compositivity (or additivity) and indecomposability of $\mathcal{Q}$ follow as in Case 1. $\square$

3.2.3. Corollary (cf. [51, Thm. 1.1]**.** *A hereditary compositive property is irreducible over $\mathbb{L}^c$ iff it is irreducible iff it is indecomposable. An additive hereditary property is irreducible over $\mathbb{L}^a$ iff it is irreducible iff it is indecomposable.*

Proof: Let $\mathcal{P}$ be hereditary compositive. If $\mathcal{P}$ is reducible over $\mathbb{L}^c$, then it is decomposable by Lemma 3.1.1; conversely, if $\mathcal{P}$ is decomposable, then by Theorem 3.2.2 it is the product of two hereditary compositive properties.

Now, if $\mathcal{P}$ is reducible over $\mathbb{L}^c$, then trivially it is reducible. Conversely, suppose $\mathcal{P}$ is the product of two arbitrary properties $\mathcal{Q}$ and $\mathcal{R}$ (not necessarily hereditary). Then $\mathcal{P} = \mathcal{P}_{\subseteq} = (\mathcal{Q} \circ \mathcal{R})_{\subseteq} = \{G + H \mid G \in \mathcal{Q}, H \in \mathcal{R}\}_{\subseteq}$. So $\mathcal{P}$ has a generating set with decomposability at least 2, and by Lemma 3.1.3, $dc(\mathcal{P}) \geq 2$. By the first part of this proof, $\mathcal{P}$ is reducible over $\mathbb{L}^c$ (although we note that there seems to be no way of specifying the hereditary compositive factors of $\mathcal{P}$ in terms of $\mathcal{Q}$ and $\mathcal{R}$).

The proof for additive hereditary properties is similar. $\square$

We can therefore talk unambiguously about irreducible hereditary compositive properties and irreducible additive hereditary properties, and these are just the indecomposable properties in $\mathbb{L}^c$ or $\mathbb{L}^a$, respectively. Similarly, properties irreducible over $\mathbb{L}$ are, in fact, irreducible (Proposition 2.1.3), but Mihók et al. showed that the following set of graphs generates a hereditary property $\mathcal{P}_1$ that is both decomposable and irreducible [51, Example 4.2]; here C_k is the cycle of length k.

$$\{C_k + C_l \mid 5 \leq k < l, k \not\equiv l \pmod 3\}.$$

The properties $\mathcal{P}_2$ and $\mathcal{Q}_i$, generated by $\{C_m \mid m \geq 5\}$ and $\{C_k \mid k \geq 5, k \not\equiv i \pmod 3\}$, respectively, are indecomposable, by Lemma 3.1.1, and thus irreducible. Yet

$$\{C_k + C_l + C_m \mid \min\{k, l, m\} \geq 5, k \not\equiv l \pmod 3\}$$

generates both $\mathcal{P}_1 \circ \mathcal{P}_2$ and $\mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \mathcal{Q}_3$. Thus hereditary properties are not uniquely factorisable into irreducible hereditary properties. However, Mihók et al. claimed that the factorisation of Theorem 3.2.2 *is* unique, giving the following proof.

> **Theorem** [51, Thm. 1.2]**.** *Let $\mathcal{R}$ be an additive hereditary property of graphs. Then the factorisation of $\mathcal{R}$ into irreducible factors is uniquely determined apart from the order of factors.*
>
> **Proof:** We use induction on $n = dc(\mathcal{R})$. If $n = 1$, the property $\mathcal{R}$ is irreducible. Let us suppose that every property with decomposability number $1 \leq k < n$ has a unique factorisation into irreducible factors, and let $\mathcal{R}$ be a property with $dc(\mathcal{R}) = n$.
>
> The structure of the factorisation of the property $\mathcal{R}$ depends on the multiplicities of the ind-parts of $\mathcal{R}$, as described in the proof of Theorem 3.2.2.
>
> This factorisation is uniquely determined, because the generators of $\mathcal{R}$ are uniquely decomposable into ind-parts.
>
> Suppose that there exists an ind-part F of $\mathcal{R}$ with multiplicity $m(F) = k < dc(\mathcal{P}) = n$. Then we consider the properties $\mathcal{Q}_F$ and $\mathcal{Q}_{\overline{F}}$ defined in Case 1 of the proof of Theorem 3.2.2. By the induction hypothesis, they are uniquely factorisable into irreducible factors. Since the generators of $\mathcal{R}$ are uniquely $(\mathcal{Q}_F, \mathcal{Q}_{\overline{F}})$-partitionable, we are done.
>
> If for every ind-part F of $\mathcal{R}$ its multiplicity $m(F)$ in $\mathcal{R}$ is equal to n, then $\mathcal{R} = \mathcal{Q}^n$ by Case 2 of Theorem 3.2.2. $\square$

The last part of this proof offers no reasons as to why $\mathcal{Q}^n$ is not also $\mathcal{A}^m$, where $\mathcal{A} \neq \mathcal{Q}$ is irreducible and, possibly $m \neq n$. Besides, $\mathcal{Q}^n$ could be $\mathcal{R}_1 \circ \cdots \circ \mathcal{R}_m$ where the $\mathcal{R}_i$'s are irreducible, but not all equal.

We note, by contrast, the approach used in the case of hom-properties. A *core* is a graph that has no homomorphism to a proper subgraph. In [46, 47, Prop. 2.4] it is established that, if $H = H_1 + \cdots + H_n$ is a core. Then

$$\to H = (\to H_1) \circ \cdots \circ (\to H_n). \qquad (\dagger)$$

Moreover [47, Theorem 1], it is not difficult to deduce that $\to H$ is irreducible (over $\mathbb{L}$) iff H is indecomposable. It therefore follows that ($\dagger$) is the unique factorisation of $\to H$ *over the class of hom-properties*, as explained in the following comment [47, p. 191]:

> Since the decomposition of a decomposable graph into the join of indecomposable graphs is unique, the factorization of reducible hom-properties into irreducible hom-properties is unique as well.

To prove that the factorisation is unique over $\mathbb{L}$, however, a more detailed proof [47, Theorem 2] is needed, showing that, if $\mathcal{P}$ and $\mathcal{Q}$ are *arbitrary* hereditary properties such that $\to H = \mathcal{P} \circ \mathcal{Q}$, then $\mathcal{P}$ and $\mathcal{Q}$ are, in fact, hom-properties, and, moreover,

$$\mathcal{P} = \to \sum_{i \in I_1} H_i \qquad \text{and} \qquad \mathcal{Q} = \to \sum_{i \in I_2} H_i$$

for some partition (I_1, I_2) of $\{1, \ldots, n\}$.

Going back to the proof of [51, Thm. 1.2], we note that even the first part of the proof does not prove uniqueness. The second and third paragraphs seem to claim that, no matter how the factorisation described in Theorem 3.2.2 is carried out, the result is always the same. If the choice of F in Case 1 is fixed, then it is true that $\mathcal{Q}_F$ and $\mathcal{Q}_{\overline{F}}$ are uniquely determined, as would be their irreducible factors, by induction. However, if some graph $F' \neq F$ is used in Case 1, then $\mathcal{Q}_{F'}$ and $\mathcal{Q}_{\overline{F}'}$ are usually different, and there is no guarantee that the collection of their irreducible properties will be the same as that obtained from $\mathcal{Q}_F$ and $\mathcal{Q}_{\overline{F}}$. In any case, nothing stops us from factoring $\mathcal{P}$ using a procedure that is completely different from that of Theorem 3.2.2.

These are not idle concerns. It is well known [5, p. 105] that unique factorisation can fail in rings. For example, in the integral domain $\mathbb{Z}[\sqrt{-3}] := \{x+y\sqrt{-3} \mid x, y \in \mathbb{Z}\}$, the elements 2, $1+\sqrt{-3}$ and $1-\sqrt{-3}$ are all irreducible — they have no factorisation into two non-unit factors. In particular, 2 has a unique factorisation, but $4 = 2^2$ does not, because we have $4 = (1+\sqrt{-3})(1-\sqrt{-3})$; so 4 is both a power of 2, and the product of distinct irreducibles. Even more disconcerting is that, in $\mathbb{Z}[\sqrt{-7}]$, 8 factors into either two or three irreducibles, since $2^3 = (1+\sqrt{-7})(1-\sqrt{-7})$.

One can also define a norm on the integral domain $\mathbb{Z}[\sqrt{d}]$, say $N_{\sqrt{d}}(a + b\sqrt{d}) := |a^2 - db^2|$, that is positive for all non-0 elements, and such that $N_{\sqrt{d}}(xy) = N_{\sqrt{d}}(x)N_{\sqrt{d}}(y)$. Thus $N_{\sqrt{-7}}(2) < N_{\sqrt{-7}}(8)$, but the unique factorisation of 2 does not imply the unique factorisation of 2^3 over $\mathbb{Z}[\sqrt{-7}]$.

In fact, Mihók et al. themselves showed in Example 4.2 of [51] that a certain hereditary (but not additive) property has factorisations with different numbers of irreducible hereditary factors: $\mathcal{P}_1 \circ \mathcal{P}_2 = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \mathcal{Q}_3$. Trivially, $\mathcal{P}_1$ and $\mathcal{P}_2$ have a unique factorisation into irreducible hereditary properties; moreover, $dc(\mathcal{P}_i) < dc(\mathcal{P}_1 \circ \mathcal{P}_2)$, $i = 1, 2$, and yet $\mathcal{P}_1 \circ \mathcal{P}_2$ is not uniquely factorisable.

It turns out that these sorts of problems do not arise in the class of hereditary compositive properties, as we show in the next section.

3.3 Unique factorisation for hereditary compositive properties

The purpose of this section is to establish that the factorisation of Theorem 3.2.2 is, in fact, unique. We know, by Theorem 3.2.2 and Corollary 3.2.3, that a property in $\mathbb{L}^c$ is irreducible iff it is indecomposable, and that a hereditary compositive property has a factorisation into indecomposable hereditary compositive properties. We therefore need to show that there is at most one factorisation into indecomposable hereditary compositive factors.

This will also show, in particular, that an additive hereditary property $\mathcal{P}$ can only have one factorisation into indecomposable additive hereditary properties. We already know that $\mathcal{P}$ has at least one such factorisation, and that additive hereditary properties are irreducible iff they are indecomposable. Unique factorisation for $\mathbb{L}^a$ will therefore follow from that for $\mathbb{L}^c$.

We do this in the following two results.

3.3.1. Theorem. *Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ be a factorisation of the hereditary compositive property $\mathcal{P}$ into indecomposable hereditary compositive properties. Then $m = dc(\mathcal{P})$.*

3.3.2. Theorem. *A hereditary compositive property $\mathcal{P}$ can have only one factorisation with exactly $dc(\mathcal{P})$ indecomposable hereditary compositive factors.*

There are two important consequences:

3.3.3. Hereditary Unique Factorisation Theorems. *A hereditary compositive property has a unique factorisation into irreducible hereditary compositive factors; an additive hereditary property has a unique factorisation into irreducible additive hereditary factors. In each case, the number of factors is exactly $dc(\mathcal{P})$.* □

3.3.4. Theorem. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be irreducible hereditary compositive properties. Then there is a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph G.*

Proof: Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. By Corollary 3.2.3 the $\mathcal{P}_i$'s are indecomposable, and by Theorem 3.3.1, $dc(\mathcal{P}) = n$. By definition of $dc(\mathcal{P})$, there is a ($\mathcal{P}$-maximal) graph G with $dc(G) = n$, and by Lemma 3.1.1, its ind-parts form its unique $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. □

Our proofs depend heavily on the following construction of a generating set for $\mathcal{P}$. Suppose $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ is a factorisation of $\mathcal{P}$ into indecomposable hereditary compositive factors, and, for each i, we are given a generating set $\mathcal{G}_i \subseteq \mathcal{M}^*(\mathcal{P}_i)$ and a graph $H_i \in \mathcal{P}_i$. By Lemmas 3.1.4 and 3.1.5, the set $\mathcal{G}_i^{\downarrow}[H_i] := \{G \in \mathcal{G}_i \mid H_i \subseteq G,\ dc(G) = 1\}$ is also a generating set for $\mathcal{P}_i$.

We set $\mathcal{G}_1^{\downarrow}[H_1] + \cdots + \mathcal{G}_m^{\downarrow}[H_m] := \{G_1 + \cdots + G_m \mid \forall\, i\ G_i \in \mathcal{G}_i^{\downarrow}[H_i]\}$.[1] This is clearly a generating set for $\mathcal{P}$, but need not consist of $\mathcal{P}$-maximal graphs (even if $m = dc(\mathcal{P})$). However, we can add edges to each graph $G_1 + \cdots + G_m$ until we get (in all possible ways) a $\mathcal{P}$-maximal graph G' (cf. Figure 3.1). Using $G \subset\!\cdot\, H$ to mean that G is a spanning subgraph of H, we

[1] Our notation extends easily to the join of any m sets: $\mathcal{G}_1 + \cdots + \mathcal{G}_m$; and to generating sets that contain several specified subgraphs: $\mathcal{G}[H_1, \ldots, H_r]$.

can now describe the generating set we want:

$$(\mathcal{G}_1[H_1] + \cdots + \mathcal{G}_m[H_m])^{\downarrow} \;\; := \;\; \{G' \in \mathcal{M}^*(\mathcal{P}) \mid dc(G') = dc(\mathcal{P}), \text{ and } \exists\, G \in \mathcal{G}_1^{\downarrow}[H_1] + \cdots + \mathcal{G}_m^{\downarrow}[H_m]\,,\; G \subset G'\}.$$

The following is immediate from the definition, and from Lemmas 3.1.1 and 3.1.5.

3.3.5. Lemma. *Let $\mathcal{G} = (\mathcal{G}_1[H_1] + \cdots + \mathcal{G}_m[H_m])^{\downarrow}$. Then:*

1. *$\mathcal{G}$ is a generating set for $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$;*
2. *if $G \in \mathcal{G}$, then $dc(G) = dc(\mathcal{P})$;*
3. *every $G \in \mathcal{G}$ is spanned by a join of m indecomposable graphs, each of which contains a different one of $H_1, \ldots, H_m$; and*
4. *if $m = dc(\mathcal{P})$ and, for each i, $\mathcal{G}_i \subseteq \mathcal{M}(\mathcal{P})$, then $\mathcal{G} \subseteq \mathcal{G}_1[H_1] + \cdots + \mathcal{G}_m[H_m]$.* □

Because we take $G' = G'_1 + \cdots + G'_{dc(\mathcal{P})} \in \mathcal{G}$ to be a *spanned* supergraph of $G = G_1 + \cdots + G_m \in \mathcal{G}_1^{\downarrow}[H_1] + \cdots + \mathcal{G}_m^{\downarrow}[H_m]$, we must have, for each i, $V(G_i) = V(\sum_{j \in J_i} G'_j)$, where $(J_1, J_2, \ldots, J_m)$ is some partition of $\{1, 2, \ldots, n\}$. That is, each of the m ind-parts of G is a spanning subgraph of a join of ind-parts from G'. We note that although $G_i \in \mathcal{P}_i$, none of the G'_j, $j \in J_i$, need be in $\mathcal{P}_i$. In particular, the crucial observation that Theorem 3.3.1 rests on is that, if $|J_i| > 1$, then $G_i \subsetneqq \sum_{j \in J_i} G'_j$, and, since G_i was $\mathcal{P}_i$-maximal, $\sum_{j \in J_i} G'_j$ is not in $\mathcal{P}_i$.

An *ordered* $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$*-partition* of G is a partition $(V_1, \ldots, V_n)$ of $V(G)$ such that, for each $i \in \{1, \ldots, n\}$, $G[V_i]$ is in $\mathcal{P}_i$. A partition is *unordered* if there is some permutation φ of $\{1, \ldots, n\}$ such that, for each $i \in \{1, \ldots, n\}$, $G[V_i]$ is in $\mathcal{P}_{\varphi(i)}$. Since there may be several such permutations, an unordered partition generally corresponds to several ordered ones. Unless specified otherwise, a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition is ordered.

We present the proof of Theorem 3.3.2 first, since it is simpler.

Proof of Theorem 3.3.2: Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_n$ be two factorisations of $\mathcal{P}$ into $n = dc(\mathcal{P})$ indecomposable hereditary compositive factors.

Label the $\mathcal{P}_i$'s inductively, beginning with $i = n$, so that, for each i, $\mathcal{P}_i$ is inclusion-wise maximal among $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i$. For each i, j such that $i > j$, if $\mathcal{P}_i \setminus \mathcal{P}_j \neq \emptyset$, then let $X_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_j$; if $\mathcal{P}_i \setminus \mathcal{P}_j = \emptyset$, then $\mathcal{P}_i = \mathcal{P}_j$ and we set $X_{i,j}$ to be the null graph. For each i, by compositivity there is an $H_{i,0} \in \mathcal{P}_i$ that contains all the $X_{i,j}$'s as subgraphs. The important point is that if $\{U_1, U_2, \ldots, U_n\}$ is an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of some graph G such that, for each $i = 1, 2, \ldots, n$, $H_{i,0} \subseteq G[U_i]$, then, by reverse induction on i starting at n, $G[U_i] \in \mathcal{P}_i$.

For each i, let $\mathcal{G}_i = \{G_{i,0}, G_{i,1}, G_{i,2}, \ldots\}$ be a generating set for $\mathcal{P}_i$. When graphs have a double subscript, we will use the second number to denote which step of our construction we are in. We start with $H_0 = H_{1,0} + \cdots + H_{n,0}$.

For each $s \geq 0$, let $H_{s+1} \in (\mathcal{G}_1[H_{1,s}, G_{1,s}] + \cdots + \mathcal{G}_n[H_{n,s}, G_{n,s}])^{\downarrow}$. Then H_{s+1} has an ind-part from each $\mathcal{G}_i[H_{i,s}, G_{i,s}]$; we label the ind-parts as $H_{1,s+1}$, $\ldots$, $H_{n,s+1}$, so that, for each i, $H_{i,0} \subseteq H_{i,1} \subseteq H_{i,2} \subseteq \cdots$

For $\mathcal{G}_i[H_{i,s}, G_{i,s}]$ to be non-empty, we must have $H_{i,s} \in \mathcal{P}_i$. We know that the $H_{i,s+1}$'s give an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of H_{s+1}. From the earlier remark, for $i = 1, 2, \ldots, n$, $H_{i,s+1} \in \mathcal{P}_i$.

By Lemma 3.1.1, the ind-parts of H_s form its (unique) unordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$-partition, so there is some permutation φ_s of $\{1, 2, \ldots, n\}$ such that, for each i, $H_{i,s} \in \mathcal{Q}_{\varphi_s(i)}$. Since there are only finitely many permutations of $\{1, 2, \ldots, n\}$, there must be some permutation φ that appears infinitely often. Now whenever $\varphi_t = \varphi$, we have $H_{i,1} \subseteq H_{i,2} \subseteq \cdots \subseteq H_{i,t} \in \mathcal{Q}_{\varphi(i)}$, so by heredity, for every $s \leq t$, $H_{i,s}$ is in $\mathcal{Q}_{\varphi(i)}$. Therefore, we can take $\varphi_s = \varphi$ for all s. By re-labelling the $\mathcal{Q}_i$'s, we can assume φ is the identity permutation, so that $H_{i,s} \in \mathcal{Q}_i$ for all i and s.

For each i and s, $G_{i,s-1} \subseteq H_{i,s}$, so that $\mathcal{H}_i := \{H_{i,1}, H_{i,2}, \ldots\}$ is a generating set for $\mathcal{P}_i$. But $\mathcal{H}_i \subseteq \mathcal{Q}_i$, so $\mathcal{P}_i = (\mathcal{H}_i)_{\subseteq} \subseteq \mathcal{Q}_i$.

By the same reasoning, there is a permutation τ such that $\mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)}$. We cannot relabel the $\mathcal{P}_i$'s as well, but if $\tau^k(i) = i$, then we have $\mathcal{P}_i \subseteq \mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)} \subseteq \mathcal{Q}_{\tau(i)} \subseteq \mathcal{P}_{\tau^2(i)} \subseteq \mathcal{Q}_{\tau^2(i)} \subseteq \cdots \subseteq \mathcal{P}_{\tau^k(i)} = \mathcal{P}_i$, so we must have equality throughout; in particular, $\mathcal{P}_i = \mathcal{Q}_i$ for each i. □

Now for the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1: Given any generating set $\mathcal{G}_i$ for $\mathcal{P}_i$, every graph in $\mathcal{G}_1 + \cdots + \mathcal{G}_m$ has decomposability m by construction. Then every graph in $(\mathcal{G}_1 + \cdots + \mathcal{G}_m)^{\downarrow} \subseteq \mathcal{M}^*(\mathcal{P})$ has decomposability at least m, so $dc(\mathcal{P}) \geq m$.

If $m < n := dc(\mathcal{P})$, and G is a $\mathcal{P}$-maximal graph with decomposability n, then, in any $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition of G, some $\mathcal{P}_i$-part is the join of two or more ind-parts. There is only a finite number of ways in which this can happen, and we will construct a sequence of generating sets, each excluding at least one of the possibilities, until we reach a contradiction.

When graphs or sets have a double subscript, we will use the second number to denote which step of our construction we are in. For each i, we start with some generating set $\mathcal{G}_i$ consisting only of indecomposable $\mathcal{P}_i$-strict graphs.

Let $H_1 \in (\mathcal{G}_1 + \cdots + \mathcal{G}_m)^{\downarrow}$; then H_1 is a join $H_{1,1} + \cdots + H_{n,1}$ of n ind-parts. In general, suppose we have graphs $H_1, H_2, \ldots, H_{k-1}$ such that, for each $s = 1, 2, \ldots, k-1$:

(a) H_s is $\mathcal{P}$-maximal;

(b) $dc(H_s) = n$, and $H_s = H_{1,s} + \cdots + H_{n,s}$;

(c) for $j = 1, \ldots, n$, $H_{j,1} \subseteq H_{j,2} \subseteq \cdots \subseteq H_{j,k-1}$; and

(d) there is a partition $(J_{1,s}, J_{2,s}, \ldots, J_{m,s})$ of $\{1, 2, \ldots, n\}$ such that

$$H'_{i,s} := \sum_{j \in J_{i,s}} H_{j,s} \text{ is in } \mathcal{P}_i.$$

Now pick $H_k \in (\mathcal{G}_1[H'_{1,(k-1)}] + \cdots + \mathcal{G}_m[H'_{m,(k-1)}])^{\downarrow}$ (see Figure 3.1). As H_k contains H_{k-1}, by Lemma 3.1.2 we can label the ind-parts of $H_k = H_{1,k} + \cdots + H_{n,k}$ so that $H_{1,(k-1)} \subseteq H_{1,k}, \ldots, H_{n,(k-1)} \subseteq H_{n,k}$. It is important to note that the indecomposable graph $G_{i,(k-1)} \in \mathcal{G}_i[H'_{i,(k-1)}]$ therefore spans $\sum_{j \in J_{i,(k-1)}} H_{j,k}$ (note the change in subscript). By Lemma 3.1.1 there is a partition $(J_{1,k}, \ldots, J_{m,k})$ of $\{1, 2, \ldots, n\}$ so that $H'_{i,k} := \sum_{j \in J_{i,k}} H_{j,k} \in P_i$.

Since there is only a finite number of partitions of $\{1, 2, \ldots, n\}$, at some step B we must end up with a partition that occurred at some previous step $A < B$. Without loss of generality, suppose that $|J_{1,A}| = r \geq 2$. Then $H'_{1,A} \in \mathcal{P}_1$; the indecomposable graph $G_{1,A} \in \mathcal{G}_1[H'_{j,A}]$ that is used in step $A+1$ spans $\sum_{j \in J_{1,A}} H_{j,(A+1)}$; this join properly contains the $\mathcal{P}_1$-maximal graph $G_{1,A}$ and therefore is not in $\mathcal{P}_1$. But, for each j, $H_{j,(A+1)} \subseteq H_{j,(A+2)} \subseteq \cdots \subseteq H_{j,B}$, and so $\sum_{j \in J_{1,A}} H_{j,(A+1)} \subseteq \sum_{j \in J_{1,A}} H_{j,B}$. But $J_{1,A} = J_{1,B}$ and $H'_{1,B} := \sum_{j \in J_{1,B}} H_{j,B} \in \mathcal{P}_1$, so $\sum_{j \in J_{1,A}} H_{j,(A+1)} \in \mathcal{P}_1$, a contradiction.

Thus we must have $|J_{i,A}| = 1$, for each $i = 1, 2, \ldots, m$, and so $m = n$. $\square$

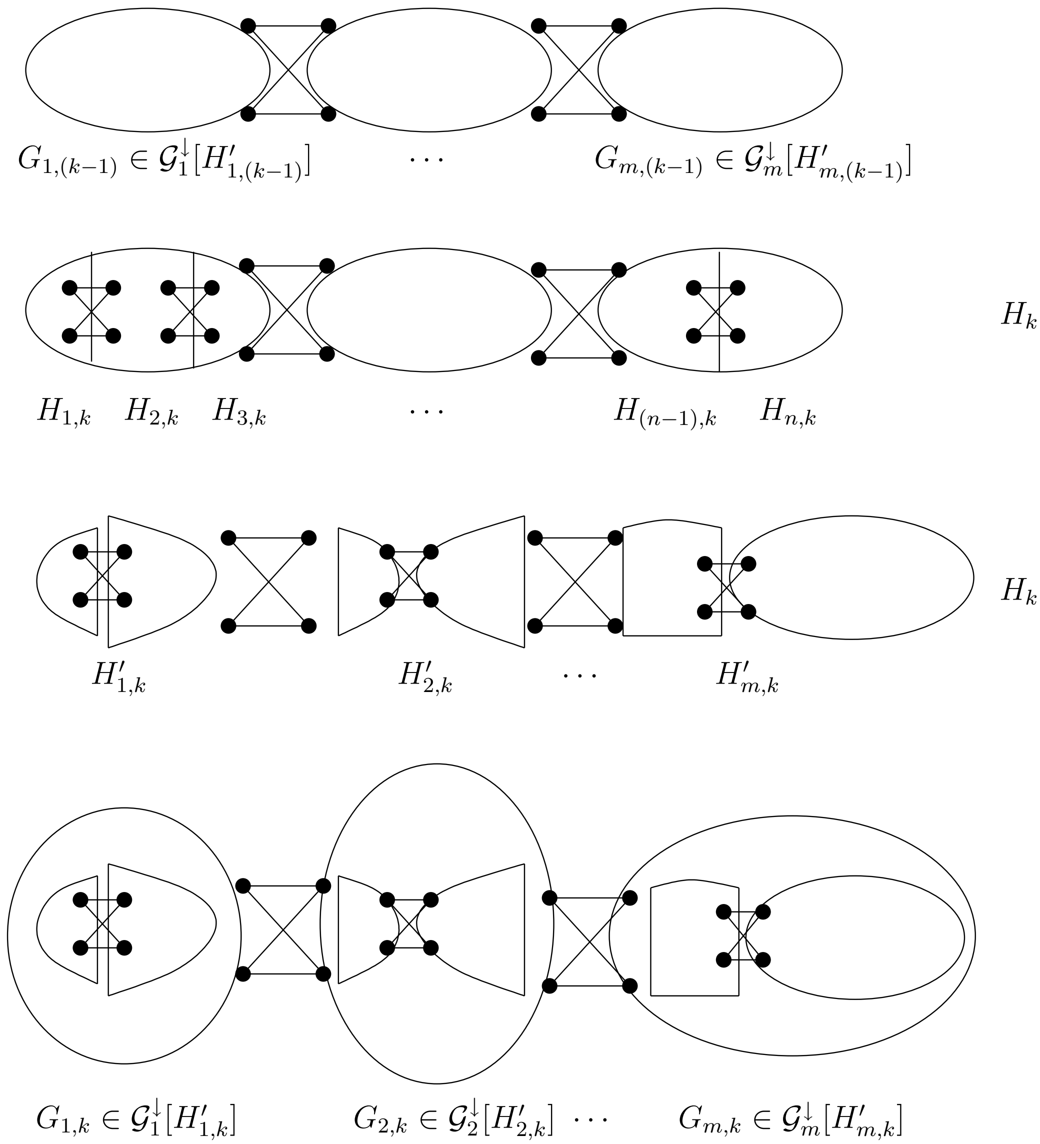


Figure 3.1: We add edges to $G_{1,(k-1)}+\cdots+G_{m,(k-1)}$ to produce a $\mathcal{P}$-maximal graph H_k. Some of the ind-parts will split into joins of smaller parts; H_k must then have a different $(\mathcal{P}_1,\ldots,\mathcal{P}_m)$-partition from that of H_{k-1}.

Chapter 4

Unique factorisation — indiscompositive properties

This chapter is the indiscompositive version of Chapter 3. In [52] Mihók generalised the factorisation of [51] to the class of additive *induced*-hereditary graph properties. In Section 4.1 we extend the concepts of [52] even further, to induced-hereditary disjoint compositive properties (for short, indiscompositive properties).

A crucial part of Mihók's work was his remarkably general construction of what he termed uniquely $\mathcal{P}$-decomposable graphs. In Section 4.2 we generalise Mihók's construction, and specify more precisely the behaviour of these graphs. The factorisation into indecomposable properties is very similar to that of Theorem 3.2.2, so we do not give it here.

Mihók claimed that this factorisation was unique, using the same argument [51] that we reproduced in Section 3.2, where we explained its shortcomings. In Section 4.3 we therefore generalise our own uniqueness proofs, to show that indiscompositive properties factor uniquely into indecomposable indiscompositive properties.

Apart from its intrinsic interest, this result has two important consequences. If $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, where the $\mathcal{P}_i$'s are irreducible, then uniquely $\mathcal{P}$-decomposable graphs turn out to be uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable (Corollary 4.3.6). The existence of such graphs was open until quite recently. Even the special case $\mathcal{P}_1 = \cdots = \mathcal{P}_n = \mathcal{Q}$, where $\mathcal{Q}$ has just one forbidden subgraph, was only solved in 1996 [7, 24, 2], so it is remarkable to have the general case proved in the space of 20 pages. Uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-

colourable graphs are, in turn, extremely useful in proving complexity results (cf. Chapter 6, [1, 31]).

Broere and Bucko [18] generalised this result to characterise the existence of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable graphs when the $\mathcal{P}_i$'s need not be irreducible (Theorem 5.3.2). Their result implies that indecomposable indiscompositive properties are actually irreducible (over $\mathbb{U}$), as we shall see in Section 5.2. It seems as if we are 'pulling ourselves up by the bootstraps' — unique factorisation into indecomposables being used to strengthen itself into unique factorisation into irreducibles.

We caution the reader that the definitions given in this chapter of "generating set", "join", "decomposability", "$\mathcal{P}$-strict" and "ind-part" differ significantly from those of Chapter 3. Unless explicitly stated, these new definitions will apply throughout the rest of the thesis, even for additive hereditary properties (that are certainly indiscompositive).

4.1 Indiscompositive properties — preliminaries

This section is the indiscompositive analogue of Section 3.1. We present the basic definitions and results from [52], along with some of our own that are relevant to disjoint compositive properties.

The smallest induced-hereditary property that contains a set $\mathcal{G}$ is denoted by $\mathcal{G}_{\leq}$. This is the induced-hereditary property *generated by* $\mathcal{G}$, or that $\mathcal{G}$ *generates*. We say that $\mathcal{G}$ is a *generating set* for $\mathcal{P}$ if $\mathcal{G}_{\leq} = \mathcal{P}$. It is easy to see that:

$$\mathcal{G}_{\leq} \;=\; \{G \mid \exists\, H \in \mathcal{G},\ G \leq H\}.$$

The $*$-*join* of n graphs $G_1, \ldots, G_n$ is the set

$$G_1 * \cdots * G_n := \{G \mid \bigcup_{i=1}^{n} G_i \subseteq G \subseteq \sum_{i=1}^{n} G_i\}$$

where $\bigcup$ and $\sum$ represent disjoint union and join, respectively. Given n sets of graphs, we define their $*$-join by

$$S_1 * \cdots * S_n := \bigcup \{G_1 * \cdots * G_n \mid \forall i, G_i \in S_i\}.$$

We note that this is just the same as $S_1 \circ \cdots \circ S_n$, but it is aesthetically pleasing to have the $*$ notation.

If G is a graph, then $s \circledast G$ denotes the set $G * G * \cdots * G$, where there are s copies of G. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be disjoint compositive properties. Mihók identified some useful necessary conditions for a partition of $V(G)$ to be a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition. Let $(W_1, \ldots, W_n)$ be a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of $G \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. Then, for each i, $G[W_i]$ is in $\mathcal{P}_i$; so, for all positive integers k, there is some graph $G_{i,k} \in k \circledast G[W_i]$ that is in $\mathcal{P}_i$, and thus

$$G_{1,k} * \cdots * G_{n,k} \subseteq \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n.$$

In particular, we can put in appropriate edges between the $G_{i,k}$'s to get a graph with k disjoint induced copies of G.

A *$\mathcal{P}$-decomposition of G with n parts* is a partition $(V_1, \ldots, V_n)$ of $V(G)$ (where, for all i, $V_i \neq \emptyset$), for which we can find a *decomposition sequence* — a sequence of graphs $G_1, G_2, \ldots$, such that, for each k:

(a) $G_k \in k \circledast G$, and

(b) $(G_k \cap V_1) * \cdots * (G_k \cap V_m) \subseteq \mathcal{P}$.

(By $G_k \cap V_i$ we mean the subgraph of G_k induced by the k copies of V_i.) When $\mathcal{P}$ is additive, we can take G_k to be kG, the disjoint union of k copies of G. If $\mathcal{P}$ is only disjoint compositive, we just have *some* fixed configuration of edges between the k copies that works. We will show, however, that there is always a "nice" configuration; in particular, we would like to have $G_1 \leq G_2 \leq \cdots$.

Once we have G_k, we can define G_{k-1} to be an induced-subgraph of G_k; however, we do not know if there is a G_{k+1} that contains G_k. Another problem is that G_k may be quite unstructured — the k disjoint induced copies of G form $\binom{k}{2}$ pairs, and the graph induced by one particular pair may look very different from that induced by any other pair. Happily, we can solve both problems simultaneously.

A *tournament* is a directed graph in which every two vertices are joined by exactly one directed edge; it is *transitive* if we can label the vertices so that $v_i \to v_j$ iff $i > j$. If A and B are subgraphs of C, we denote $C[V(A) \cup V(B)]$ by $C[A : B]$, or just $A : B$ if C is understood from the context.

A *tournament $\mathcal{P}$-decomposition* is a $\mathcal{P}$-decomposition for which we can find a *tournament sequence* — a decomposition sequence such that:

(c) $G_1 \leq G_2 \leq \cdots$, and

(d) for each k, we can order the k induced copies of G in G_k as $G^1, \ldots, G^k$ so that, whenever $p < q$ and $r < s$, $G^p : G^q$ and $G^r : G^s$ are essentially the same as *labeled* subgraphs.

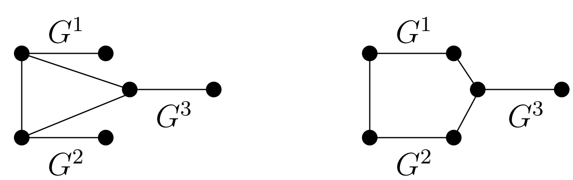
More precisely, what we mean by (d) is the following. Let the vertices of G be $v_1, \ldots, v_t$. Then we can label the vertices of G_k as $v_1^i, \ldots, v_t^i$, $1 \leq i \leq k$, so that

(i) the mapping $\alpha : v_j^i \to v_j$ is an isomorphism from G^i to G; and

(ii) for each $p < q$ and $r < s$, the mapping defined by $\phi(v_j^p) = v_j^r$ and $\phi(v_j^q) = v_j^s$ is an isomorphism from $G^p : G^q$ to $G^r : G^s$.

Note that, with this labelling, if $k \leq \ell$, and $U \subseteq V(G_k)$, then $G_k[U]$ and $G_\ell[U]$ are the same labeled graph; in particular, we do not need to specify whether G^p is a subgraph of G_k or G_ℓ, and there is no ambiguity in the notation $G^p : G^q$.

Condition (d) is quite restrictive — we could have $G'_k \in k \circledast G$, with $V(G_k) = V(G'_k)$, such that $G_k \ncong G'_k$, and yet have $G_k[G^p : G^q]$ and $G'_k[G^p : G^q]$ isomorphic as *unlabeled* graphs), for all $1 \leq p < q \leq k$, (see Figure 4.1). On the other hand, if $G_k[G^p : G^q]$ and $G'_k[G^p : H^q]$ are always isomorphic as *labeled* graphs, then clearly $G_k \cong G'_k$.



G_3 $\qquad\qquad$ G'_3

Figure 4.1: G_3 and G'_3 are not isomorphic.

The G_k's that form a decomposition sequence need not be a tournament sequence, but we will show that we can always find (induced-subgraphs of) a subsequence of the G_k's that satisfy conditions (c) and (d). This fact will be important in the proof of Theorem 4.3.3, as we will choose sequences with particular properties, knowing that from there we can find tournament sequences with the same properties.

4.1.1. Proposition. *A $\mathcal{P}$-decomposition is a tournament $\mathcal{P}$-decomposition.*

Proof: Let $d = (V_1, \dots, V_n)$ be a $\mathcal{P}$-decomposition of some graph G. The crucial point is that G is a finite graph, so there are only finitely many graphs in $2 \circledast G$; there are many more *labeled* graphs, but it is still a finite number, say M.

Let $G_1, G_2, G_3, \dots$ be a decomposition sequence for d. Let the vertices of G be $v_1, \dots, v_t$. We can label the vertices of G_k as $v_1^i, \dots, v_t^i$, $1 \le i \le k$, so that, for each i, $G_k^i := G_k[v_1^i, \dots, v_t^i]$ is a labeled copy of G.

For each k, define a transitive tournament T_k that has $G_k^1, \dots, G_k^k$ as vertices, with $G_k^p \to G_k^q$ iff $p < q$. Colour each directed edge $G_k^p \to G_k^q$ with colour c_m, $1 \le m \le M$, according to which labeled graph is induced by $G_k^p : G_k^q$. The edge-coloured tournament T_k thus encodes the structure of G_k.

By Ramsey's theorem [60, Thm. 8.3.7], for any r there is some T_{k_r} that must contain a monochromatic tournament of order r. Since there are only finitely many colours, there must be some colour c_ℓ for which there are infinitely many monochromatic tournaments coloured c_ℓ; let these appear in $T_{k'_1}, T_{k'_2}, T_{k'_3}, \dots$. Thus, for each $j \ge 1$, $T_{k'_j}$ contains a transitive tournament on at least j vertices, whose edges are all coloured c_ℓ. Then the graphs $G_{k'_1}, G_{k'_2}, G_{k'_3}, \dots$ contain induced-subgraphs $G'_1 \le G'_2 \le G'_3 \le \cdots$, with each $G'_k \in k \circledast G$, that satisfy condition (d). $\square$

4.1.2. Lemma. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$. If the $\mathcal{P}_i$'s are induced-hereditary disjoint compositive, then any $(\mathcal{P}_1, \dots, \mathcal{P}_m)$-partition of a graph G is a $\mathcal{P}$-decomposition of G; every graph in $\mathcal{P}$ with at least m vertices has a $(\mathcal{P}_1, \dots, \mathcal{P}_m)$-partition with all m parts non-empty.* $\square$

The *$\mathcal{P}$-decomposability number $dec_\mathcal{P}(G)$* of G is the maximum number of parts in a $\mathcal{P}$-decomposition of G; if $G \notin \mathcal{P}$, then we put $dec_\mathcal{P}(G) = 0$. If $G \in \mathcal{P}$, then $V(G)$ is a $\mathcal{P}$-decomposition; therefore $G \in \mathcal{P}$ if and only if $dec_\mathcal{P}(G) \ge 1$. The graph G is *$\mathcal{P}$-decomposable* if $dec_\mathcal{P}(G) > 1$. If $\mathcal{P}$ is the product of two induced-hereditary disjoint compositive properties, then *every* graph in $\mathcal{P}$ with at least two vertices is $\mathcal{P}$-decomposable.

The *$\mathcal{P}$-decomposability number* of a set $\mathcal{G}$ is

$$dec_\mathcal{P}(\mathcal{G}) := \min\{dec_\mathcal{P}(G) \mid G \in \mathcal{G}\}.$$

A graph G is *$\mathcal{P}$-strict* if $G \in \mathcal{P}$ but $G * K_1 \notin \mathcal{P}$; the set of $\mathcal{P}$-strict graphs is $\mathbf{S}(\mathcal{P})$. The *decomposability number $dec(\mathcal{P})$* of $\mathcal{P}$ is $dec_\mathcal{P}(\mathbf{S}(\mathcal{P}))$.

Let $f(\mathcal{P}) := \min\{|V(F)| \mid F \notin \mathcal{P}\}$; then $G * K_1 * \cdots * K_1 \nsubseteq \mathcal{P}$, where the $*$ operation is repeated $f(\mathcal{P})$ times. Thus, every $G \in \mathcal{P}$ is an induced-subgraph of some $\mathcal{P}$-strict graph (with fewer than $|V(G)| + f(\mathcal{P})$ vertices), and so $\mathbf{S}(\mathcal{P})_{\leq} = \mathcal{P}$. Similarly, $dec_{\mathcal{P}}(G) < f(\mathcal{P})$.

The following are the analogues of Lemmas 3.1.1 – 3.1.5; their proofs are essentially the same.

4.1.3. Lemma. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be induced-hereditary properties of graphs, and let G be a $(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m)$-strict graph. Then, for every $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $(V_1, \ldots, V_m)$ of $V(G)$, $G[V_i]$ is $\mathcal{P}_i$-strict (and, in particular, non-empty).*

Proof: If, say, $G[V_1] * K_1 \subseteq \mathcal{P}_1$, then $G * K_1 = (G[V_1] * K_1) * G[V_2] * \cdots * G[V_m] \subseteq \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$. □

It follows that $dec(\mathcal{A} \circ \mathcal{B}) \geq dec(\mathcal{A}) + dec(\mathcal{B})$, and thus any factorisation of an indiscompositive property $\mathcal{P}$ has at most $dec(\mathcal{P})$ irreducible indiscompositive factors.

4.1.4. Lemma [52]**.** *Let G be a $\mathcal{P}$-strict graph, for some indiscompositive property $\mathcal{P}$, and suppose that $G \leq G' \in \mathcal{P}$. Then G' is $\mathcal{P}$-strict, and $dec_{\mathcal{P}}(G) \geq dec_{\mathcal{P}}(G')$.* □

4.1.5. Lemma [52]**.** *If $\mathcal{G}$ generates the induced-hereditary property $\mathcal{P}$, then $dec_{\mathcal{P}}(\mathcal{G}) \leq dec_{\mathcal{P}}(\mathbf{S}(\mathcal{P}))$, with equality if $\mathcal{G} \subseteq \mathbf{S}(\mathcal{P})$.* □

For $\mathcal{G} \subseteq \mathcal{P}$, and $H \in \mathcal{P}$, let $\mathcal{G}[H] := \{G \in \mathcal{G} \mid H \leq G\}$. We have already stated the next result as Theorem 2.2.1(B).

4.1.6. Lemma [52]**.** *Let $\mathcal{G}$ generate the induced-hereditary compositive property $\mathcal{P}$, and let H be a graph in $\mathcal{P}$. Then $\mathcal{G}[H]$ also generates $\mathcal{P}$.* □

For a set $\mathcal{G}$, let $\mathcal{G}^{\downarrow} := \{G \in \mathcal{G} \mid G \in \mathbf{S}(\mathcal{P}),\ dec_{\mathcal{P}}(G) = dec(\mathcal{P})\}$. The following is a simple consequence of Lemmas 4.1.4 and 4.1.6.

4.1.7. Lemma [52]**.** *If $\mathcal{G}$ generates the induced-hereditary compositive property $\mathcal{P}$, then so does $\mathcal{G}^{\downarrow}$.* □

4.2 Uniquely $\mathcal{P}$-decomposable graphs

When dealing with hereditary compositive properties, it was very important that, if $G = G_1 + \cdots + G_m$, then each G_i is the join of ind-parts — the partition into ind-parts "respected" the partition into G_i's. In this section we prove analogous results for $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graphs with minimum decomposability, which allows us to generalise Theorem 3.3.1.

The required results are corollaries of Theorem 4.2.2, essentially due to Mihók. He actually proved the case where $m = n$ (Corollary 4.2.4), but, with the appropriate concepts that we introduce below, very little modification is needed to establish the result we need for $m \leq n$; we follow Mihók's proof and notation rather closely.

A graph $G \in \mathcal{P}$ is *uniquely* $\mathcal{P}$*-decomposable* if there is only one $\mathcal{P}$-decomposition of G with $dec_{\mathcal{P}}(G)$ parts. Equivalently, G is either $\mathcal{P}$-indecomposable, or has exactly one $\mathcal{P}$-decomposition with n parts, for some $n \geq 2$; in the latter case, n must be $dec_{\mathcal{P}}(G)$, as any decomposition with $n+1$ parts would give rise to $\binom{n+1}{2}$ decompositions with n parts.

If $(V_1, \ldots, V_n)$ is the unique $\mathcal{P}$-decomposition of G, we call the graphs $G[V_1]$, $\ldots$, $G[V_n]$ its *ind-parts* (although we note that they are themselves usually $\mathcal{P}$-decomposable).

4.2.1. Lemma [52]**.** *Let G be a graph in $\mathbf{S}(\mathcal{P})$ with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and suppose that G has a unique $\mathcal{P}$-decomposition $(V_1, \ldots, V_{dec(\mathcal{P})})$ with $dec(\mathcal{P})$ parts. If $G \leq H$, then $H \in \mathbf{S}(\mathcal{P})$, $dec_{\mathcal{P}}(H) = dec(\mathcal{P})$, and, for any $\mathcal{P}$-decomposition $(W_1, \ldots, W_{dec(\mathcal{P})})$ of H, we can relabel the W_i's so that, for all i, $W_i \cap V(G) = V_i$.* □

We recall that if G is a graph, then $s \circledast G$ denotes the set $G * G * \cdots * G$, where there are s copies of G. For $G^* \in s \circledast G$, denote the copies of G by $G^1, \ldots, G^s$. Let d_0 be a partition of $V(G)$, and d a partition of $V(G^*)$. The *extension of* d_0 to G^* is the partition obtained by repeating d_0 on each copy of G. The partition of $V(G^k)$ *induced by* d is denoted $d|G^k$ (so if d is the extension of d_0, then $d|G^k = d_0$ for all k).

A partition $d_1 = (V_1, V_2, \ldots, V_n)$ of $V(G)$ *respects* a partition $d_0 = (U_1, U_2, \ldots, U_m)$ if each V_i is contained in some U_j. This means that each U_j is a disjoint[1] union of V_i's, so d_0 is just a coarser version of d_1.

A partition $d = (V_1, \ldots, V_n)$ of $V(G^*)$ *respects* d_0 *uniformly* if it respects the extension of d_0. More formally, for each V_i, there is a U_j such that, for every G^k, $V_i \cap V(G^k) \subseteq U_j$. It is possible for each $d|G^k$ to respect d_0, without d respecting d_0 uniformly.

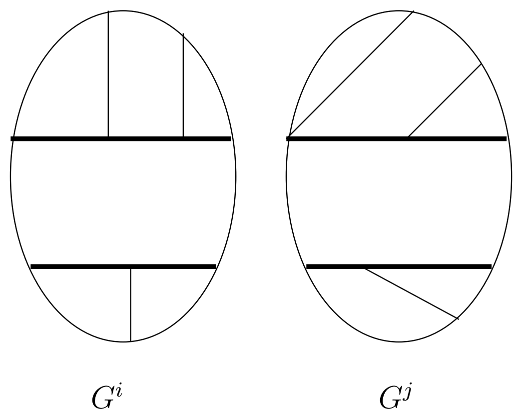


Figure 4.2: d (vertical lines) respects d_0 (horizontal lines) uniformly.

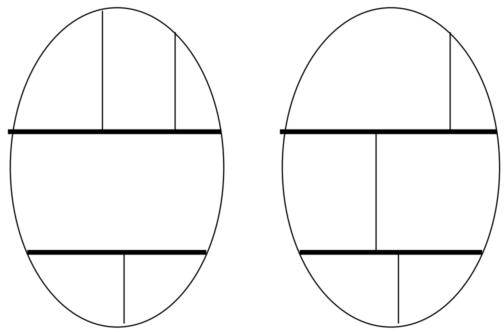
If G is uniquely $\mathcal{P}$-decomposable, its ind-parts *respect* d_0 if its unique $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G)$ parts respects d_0. If $G^* \in s \circledast G$ (for some s) is uniquely $\mathcal{P}$-decomposable, its ind-parts *respect* d_0 *uniformly* if G^*'s unique $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G^*)$ parts respects d_0 uniformly.

We may sometimes write $G^i \cap U_x$ (or just U_x when it is clear we are referring to G^i) to mean $V(G^i) \cap U_x$, and $G^* \cap U_x$ (or just U_x, when it is clear from the context) to mean $\bigcup_i (G^i \cap U_x)$.

We note that, just after proving Theorem 4.2.2, we illustrate Mihók's construction for the case $\mathcal{P} = \mathcal{K} \circ \mathcal{L}$, where $\mathcal{K} = \{\text{complete graphs}\}$ and $\mathcal{L} = \{\text{line graphs}\}$. Our intention is for the reader to refer to the concrete examples while going through the proof.

4.2.2. Theorem. *Let $\mathcal{P}$ be an indiscompositive property, G a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = n$, and $d_0 = (U_1, U_2, \ldots, U_m)$ a fixed $\mathcal{P}$-decomposition*

[1] $G[U_j]$ need not be a disjoint union of $G[V_i]$'s, as there will usually be edges between the $G[V_i]$'s.



G^i G^j

Figure 4.3: d (vertical lines) respects d_0 (horizontal lines) on both G^i and G^j, but not uniformly.

of G. Then there is a $\mathcal{P}$-strict graph $G^* \in s \circledast G$ (for some s) such that any $\mathcal{P}$-decomposition of G^* with n parts respects d_0 uniformly; moreover, the extension of d_0 is a $\mathcal{P}$-decomposition of G^*.

Proof: Let $d_k = (V_{k,1}, V_{k,2}, \ldots, V_{k,n}),\ k = 1, \ldots, r$, be the $\mathcal{P}$-decompositions of G with n parts which do not respect d_0. Since G is a finite graph, r is a nonnegative integer. If $r = 0$, take $G^* = G$; otherwise we will construct a graph $G^* = G^*(r) \in s \circledast G$ as above, denoting the s copies of G by $G^1, \ldots, G^s$.

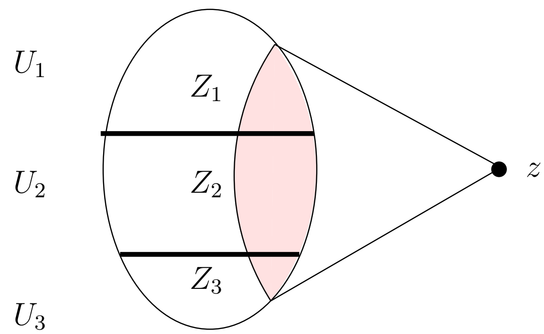
If the resulting G^* has a $\mathcal{P}$-decomposition d with n parts, then, since G is $\mathcal{P}$-strict, $d|G^i$ will also have n parts. Since $\mathcal{P}$ is indiscompositive, we know that, for any s, we can find a graph $H_s \in s \circledast G$ that is in $\mathcal{P}$; the aim of the construction is to put new edges $E^* = E^*(r)$ in an appropriate H_s, to exclude the possibility that $d|G^i = d_j$, for any $1 \leq i \leq s, 1 \leq j \leq r$.

When we say that we 'put' (rather than 'add') edges E' between two sets of vertices A and B, we mean that we take out all the edges that already existed between A and B, and replace them with E'. We will only put edges between $G^i \cap U_x$ and $G^j \cap U_y$, where $i \neq j$ and $x \neq y$; this will ensure that the extension of d_0 is a $\mathcal{P}$-decomposition of G^*, and, thus, that $G^* \in \mathcal{P}$.

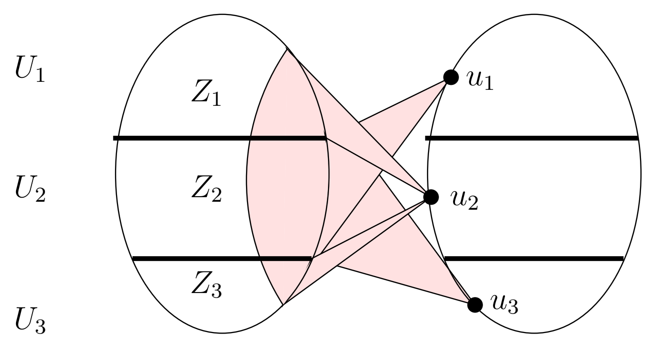
We shall need Proposition 4.1.1. This result guarantees that we can find a tournament sequence $G_1 \leq G_2 \leq \cdots$ for d_0. Thus, for each k, $G_k \in k \circledast G$ and $(G_k \cap U_1) * \cdots * (G_k \cap U_m) \subseteq \mathcal{P}$; moreover, we can label the vertices, and order the k induced copies of G as $G^1, \ldots, G^k$ so that, whenever $p < q$ and

$r < s$, $G^p : G^q$ and $G^r : G^s$ are essentially the same as labeled subgraphs.

We shall use two constructions.



$F \in G * K_1$



$G^i \quad \Rightarrow \quad G^j$

Figure 4.4: The graph F, and three sample vertices in $G^i \Rightarrow G^j$. The u_y's might also be adjacent to some vertices in Z_y.

Construction 1. $G^i \Rightarrow G^j$.

This is a graph in $2\circledast G$ such that, if d is a $\mathcal{P}$-decomposition of $G^i \Rightarrow G^j$ and $d|G^i$ respects d_0, then $d|G^j$ respects d_0; moreover, d respects d_0 uniformly on $G^i \Rightarrow G^j$. (We comment that this corrects a minor error in [52]. Mihók was independently aware of both the error and its correction.)

Since G is $\mathcal{P}$-strict, there is a graph $F \in (G * K_1) \setminus \mathcal{P}$. Let $N_F(z)$ be the neighbours in G of $z \in V(K_1)$. For $y = 1, 2, \ldots, m$, let Z_y denote $U_y \cap N_F(z)$.

We now take the graph G_2 from the tournament sequence for d_0, with two copies G^1 and G^2 of G; re-label these as G^i and G^j, if $i < j$, or G^j and G^i, if $j < i$. We form $G^i \Rightarrow G^j$ by putting edges between $U_x \cap G^j$ and $U_y \cap G^i$, $1 \leq x \neq y \leq m$, so that every vertex of $U_x \cap G^j$ is adjacent precisely to the vertices of $Z_y \cap G^i$ (see Figure 4.4).

For contradiction, let $d = (V_1, V_2, \ldots, V_n)$ be a $\mathcal{P}$-decomposition of $(G^i \Rightarrow G^j)$ such that $d|G^i$ respects d_0, but $d|G^j$ does not respect d_0 (or at least, not in the same manner, i.e., d does not respect d_0 uniformly). Then there is a k such that $V_k \cap G^i \subseteq U_y$, but some $v \in V_k \cap G^j$ belongs to U_x, $x \neq y$.

Since d is a $\mathcal{P}$-decomposition, we can put any edges we like between v and vertices not in V_k, and still remain in $\mathcal{P}$. In particular, we can put edges between v and $U_x \cap G^i$ so that v is now adjacent to the vertices of $Z_x \cap G^i$ (and to no other vertices of $U_x \cap G^i$). But then v and G^i induce a subgraph isomorphic to F, a contradiction.

Construction 2. $m.k_t \bullet G$.

For a $\mathcal{P}$-decomposition $d_t = (V_{1,t}, V_{2,t}, \ldots, V_{dec_{\mathcal{P}}(G),t})$ of G that does not respect d_0, $m.k_t \bullet G$ is a graph constructed from G_{mk_t} of the tournament sequence, such that the extension of d_t is *not* a $\mathcal{P}$-decomposition of $k_t \bullet G$. In other words, in any $\mathcal{P}$-decomposition $d = (W_1, W_2, \ldots, W_{dec_{\mathcal{P}}(G)})$ of $m.k_t \bullet G$, there is an induced copy G^i of G for which $d|G^i \neq d_t$. We also require that $m.k_t \bullet G$ be in $m.k_t \circledast G$ (see Fig.s 4.5 and 4.6).

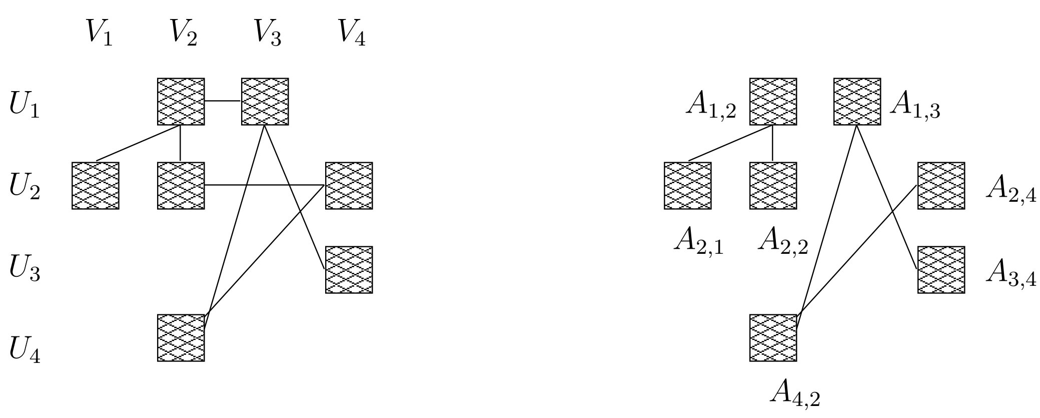


Figure 4.5: The graphs $H \notin \mathcal{P}$ and $\widetilde{H} \in \mathcal{P}$ obtained from G_{k_t}.

Let $n = dec_{\mathcal{P}}(G)$, and let $A_{i,j}(t)$ denote $U_i \cap V_{j,t}, 1 \leq i \leq m, 1 \leq j \leq n$. Since d_t does not respect d_0, at least $n+1$ sets $A_{i,j}(t)$ are nonempty. Because

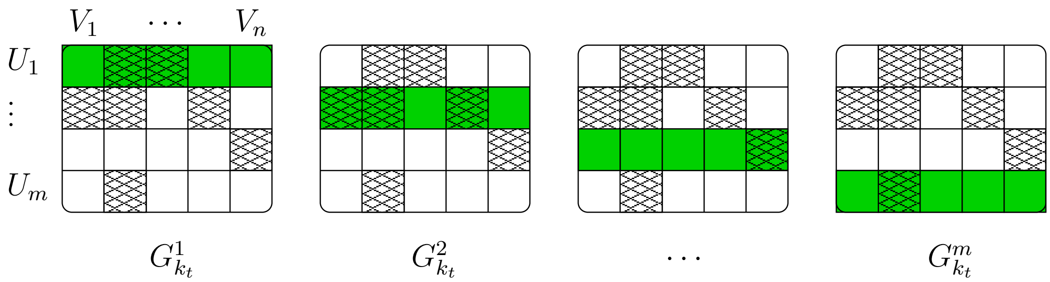


Figure 4.6: $m.k_t \bullet G$ — we only put edges between the m shaded strips.

$dec_{\mathcal{P}}(G) = n$, there exists a positive integer k_t such that, for any graph $H \in k_t \circledast G$, $(H \cap A_{1,1}(t)) * (H \cap A_{1,2}(t)) * \cdots * (H \cap A_{m,n}(t)) \not\subseteq \mathcal{P}$. In particular, for $H = G_{k_t}$ there is a graph $F_t \in (H \cap A_{1,1}(t)) * (H \cap A_{1,2}(t)) * \cdots * (H \cap A_{m,n}(t)) \setminus \mathcal{P}$.

Suppose that in H we put the same edges between $H \cap U_x$ and $H \cap U_y$ as there are between $F_t \cap U_x$ and $F_t \cap U_y$, for all $x \neq y$; the U_x's still form a $\mathcal{P}$-decomposition of the resulting graph $\widetilde{H}$, so it is in $\mathcal{P}$. If the extension of d_t were also a $\mathcal{P}$-decomposition of $\widetilde{H}$, we could obtain F_t immediately by putting the same edges between $\widetilde{H} \cap V_{i,t}$ and $\widetilde{H} \cap V_{j,t}$ as there are between $F_t \cap V_{i,t}$ and $F_t \cap V_{j,t}$, for all $i \neq j$. The only problem is that $\widetilde{H}$ does not contain k_t disjoint copies of G, as we altered edges *inside* the copies of G.

So instead we take the graph $H' = G_{mk_t}$ from the tournament sequence; this contains m disjoint induced copies of G_{k_t}, say $G^1_{k_t}, \ldots, G^m_{k_t}$, and furthermore $(H' \cap U_1) * \cdots * (H' \cap U_m) \subseteq \mathcal{P}$. In H' put the same edges between $G^x_{k_t} \cap U_x$ and $G^y_{k_t} \cap U_y$ as there are between $F_t \cap U_x$ and $F_t \cap U_y$, for all $x \neq y$; the resulting graph is $m \bullet k_t G$.

Now $G^1_{k_t} \cap U_1, \ldots, G^m_{k_t} \cap U_m$ together form a copy of $\widetilde{H}$ in $m \bullet k_t G$. Suppose $H'' = m \bullet k_t G$ has a $\mathcal{P}$-decomposition $d = (W_1, W_2, \ldots, W_n)$ such that, for every one of the mk_t induced copies G^i of G, $d|G^i = d_t$; then we can obtain F_t as an induced subgraph of a graph in $H''[W_1] * H''[W_2] * \cdots * H''[W_n]$ (by changing edges in the copy of $\widetilde{H}$ as explained above).

We point out that H'' is obtained from G_{mk_t} by changing only edges between U_x and U_y, for $x \neq y$.

We now construct G^* as follows. First let $G(1) := m.k_1 \bullet G$. For $1 < \ell \leq r$, construct $G(\ell)$ by taking the graph $G_{\kappa_\ell} \in \kappa_\ell \circledast G$ from the tournament sequence, where $\kappa_\ell = m.k_\ell.m.k_{\ell-1}.\cdots.m.k_1$. Put edges between $G_{\kappa_\ell} \cap U_x$ and $G_{\kappa_\ell} \cap U_y$ to give mk_ℓ disjoint copies $G(\ell-1)^1, \ldots, G(\ell-1)^{mk_\ell}$ of $G(\ell-1)$. We can do this since G_{κ_ℓ} contains mk_ℓ disjoint copies of $G_{\kappa_{\ell-1}}$ (because it comes from the tournament sequence); and because $G(\ell-1)$ is obtained from $G_{\kappa_{\ell-1}}$ by changing only edges between U_x and U_y, for $x \neq y$.

For each copy of G in $G(\ell-1)^i$ and each copy of G in $G(\ell-1)^j$, we put the edges between them that are between the i^{th} and j^{th} copies of G in $m.k_\ell \bullet G$. Again, we can do this because $m.k_\ell \bullet G$ is obtained from G_{mk_ℓ} by changing only edges between U_x and U_y, for $x \neq y$. (See Figure 4.7.)

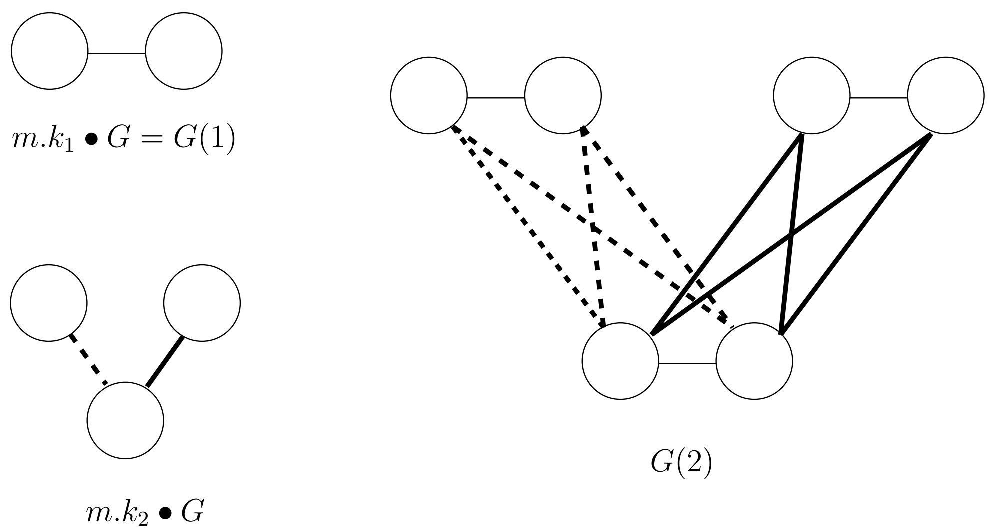


Figure 4.7: Constructing $G(2)$ from $G(1)$ and $m.k_2 \bullet G$.

Finally, we take G_{κ_r+2} from our tournament sequence, with copies $G^0, G^1, G^2, \ldots, G^{\kappa_r}, G^{\kappa_r+1}$ of G. We put edges between U_x and U_y so that:
(a) $G^1, \ldots, G^{\kappa_r}$ induce a copy of $G(r)$;
(b) for each $i \in \{1, 2, \ldots, \kappa_r\}$, G^0 and G^i induce a copy of $G^0 \Rightarrow G^i$;
(c) for each $i \in \{1, 2, \ldots, \kappa_r\}$, G^i and G^{κ_r+1} induce a copy of $G^i \Rightarrow G^{\kappa_r+1}$; and
(d) G^{κ_r+1} and G^0 induce $G^{\kappa_r+1} \Rightarrow G^0$.

Let d be a $\mathcal{P}$-decomposition of G^* with n parts (it might be that none exists, in which case we are done). For $1 \leq \ell \leq r$, if every copy of $G(\ell-1)$ in $G(\ell)$ contains a copy of G for which $d|G = d_\ell$, then we would have mk_ℓ such copies of G inducing a copy of $mk_\ell \bullet G$, which we know is impossible.

So by induction from r to 1, there is a copy G^p of G for which $d|G^p$ is none of $d_1, d_2, \ldots, d_r$. Thus, $d|G^p$ respects d_0. But $G^p \Rightarrow G^{\kappa_r+1}$ is an induced subgraph of G^*, so $d|G^{\kappa_r+1} = d_0$ (and in fact d respects d_0 uniformly on these two copies of G). Similarly, $d|G^0$ respects d_0 and, again in the same way, d respects d_0 uniformly on *all* κ_{r+2} disjoint copies of G, as required. $\square$

As promised, we now show how the construction of Theorem 4.2.2 works for the case $\mathcal{P} = \mathcal{K} \circ \mathcal{L}$, where $\mathcal{K} = \{\text{complete graphs}\}$ and $\mathcal{L} = \{\text{line graphs}\}$. We will consider the graph $G = K_{1,4} \in K_1 * \overline{K_4}$. Since $K_{2,4} \in K_{1,4} * K_1$ is not in $\mathcal{P}$, G is $\mathcal{P}$-strict. Label the vertices as $v_0, \ldots, v_4$, where v_0 is the vertex of degree 4. Since $K_{1,3}$ is not a line graph, in any $(\mathcal{K}, \mathcal{L})$-partition the $\mathcal{K}$-part cannot be empty, but must induce either a K_1 or K_2. There is one partition, $d_0 = (U_1, U_2)$, with a K_1 in the $\mathcal{K}$-part, and partitions $d_1, \ldots, d_4$ with a K_2 in the $\mathcal{K}$-part (see Figure 4.8). Here $U_1 = \{v_0\}$ and $U_2 = \{v_1, \ldots, v_4\}$, while, for $1 \le t \le 4$, $d_t = (V_{1,t}, V_{2,t})$, where $V_{1,t} = \{v_0, v_t\}$ and $V_{2,t} = U_2 \setminus \{v_t\}$.

Even on a small graph like G, it would be tedious to check that there are no other $\mathcal{P}$-decompositions; we will therefore refer to Corollary 4.2.5, that shows that every $\mathcal{P}$-decomposition with $dec_{\mathcal{P}}(G)$ parts is a $(\mathcal{K}, \mathcal{L})$-partition.

Therefore $dec_{\mathcal{P}}(G) = 2$, and in our example we have $m = n = 2$. Because $m = n$, the graph we will construct will actually be uniquely $\mathcal{P}$-decomposable (cf. Corollary 4.2.4) — any $\mathcal{P}$-decomposition with two parts will respect d_0 uniformly, so it will have to be the extension of d_0.

For our tournament sequence, we will take $G_r \in r \circledast G$, which we obtain from r disjoint copies of G by adding edges to make the copies of v_0 induce a clique. This has a $(\mathcal{K}, \mathcal{L})$-partition, where the $\mathcal{K}$-part induces K_r and the $\mathcal{L}$-part induces $\overline{K_{4r}}$.

For Construction 1, we take $F = K_{2,4} \in (K_{1,4} * K_1) \setminus \mathcal{P}$. Then $Z_1 = \emptyset$ and $Z_2 = \{v_1, \ldots, v_4\}$. The graph $G^i \Rightarrow G^j$ is shown in Figure 4.8. Suppose we have a $\mathcal{P}$-decomposition $d = (U_1, U_2)$ such that $U_1 \cap G^i = \{v_{0,i}\}$ and $U_2 \cap G^i = \{v_{1,i}, \ldots, v_{4,i}\}$. Then, if $v_{0,j} \notin U_1$, we could remove the edge $v_{0,i}v_{0,j}$ so that G^i and $v_{0,j}$ together form $K_{2,4}$, a contradiction. For $1 \le k \le 4$, if we have $v_{k,j}$ in U_1, then we could add edges between $v_{k,j}$ and $v_{1,i}, \ldots, v_{4,i}$, so that G^i and $v_{k,j}$ form a $K_{2,4}$. So if d respects d_0 on G^i, it must respect it uniformly on $(G^i \Rightarrow G^j)$.

For Construction 2, we consider the decomposition $d_t = d_1$ that does not respect d_0. Then $A_{1,1}(t) = \{v_0\}, A_{1,2}(t) = \emptyset, A_{2,1}(t) = \{v_1\}, A_{2,2}(t) = \{v_2, v_3, v_4\}$ (see Figure 4.9). We will abuse notation, letting $A_{i,j}(t)$ denote also the extension of $A_{i,j}(t)$. We have $3 > dec_{\mathcal{P}}(G)$ non-empty parts, and,

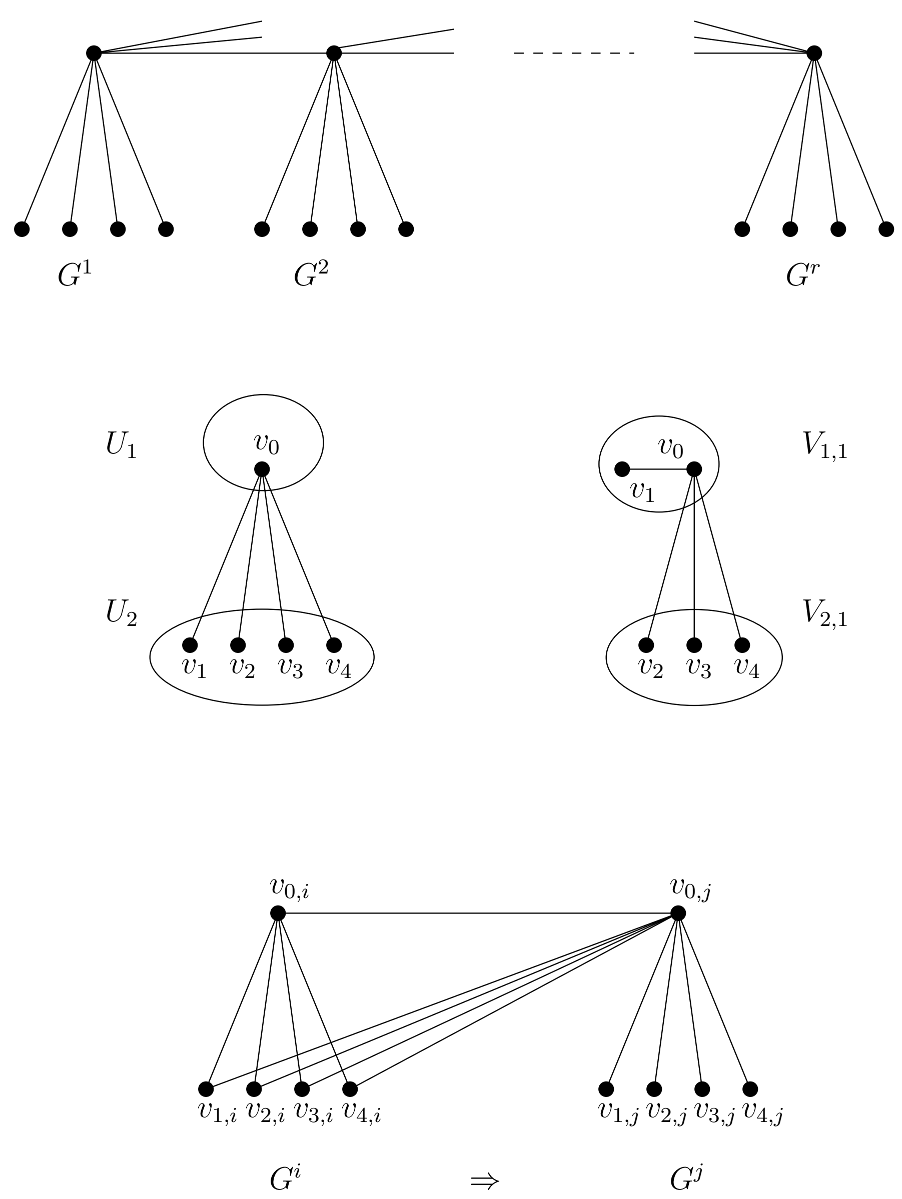


Figure 4.8: The graph G_r from the tournament sequence for G, the $\mathcal{P}$-decompositions d_0 and d_1, and the graph $G^i \Rightarrow G^j$.

in fact, taking $k_t = 2$ and $H = G_{k_t}$ from the tournament sequence, there is $F_t \in (H \cap A_{1,1}(t)) * (H \cap A_{1,2}(t)) * (H \cap A_{2,1}(t)) * (H \cap A_{2,2}(t)) \setminus \mathcal{P}$. Although F_t happens to contain a copy of $F \in G * K_1$ (which is why $F_t \notin \mathcal{P}$), we obtain it not just by adding edges between the two copies of G, but by adding edges (indicated in bold) between the $A_{p,q}$'s; we could also have deleted edges between the $A_{p,q}$'s if necessary.

Since $v_{0,1}v_{2,2}$ is the only change that we make between U_1 and U_2, the graph $\widetilde{H}$ is just $G_{k_t} + v_{0,1}v_{2,2}$. The graph $\widetilde{H}$ is still in $\mathcal{P}$, with $\mathcal{P}$-decomposition (U_1, U_2), but if (V_1, V_2) were also a $\mathcal{P}$-decomposition, then we could obtain F_t, a contradiction.

We now take $m = 2$ copies of G_{k_t}, and produce $m.k_t \bullet G$ by changing edges between U_1 and U_2 so that $U_1 \cap G^1$ and $U_2 \cap G^2$ induce a copy of $\widetilde{H}$. A more general sketch is provided in Figure 4.6, as are sketches of the construction of G^* (Figure 4.7). The reason we use $m.k_t \bullet G$, rather than $\widetilde{H}$, is to ensure that we have a graph in $r \circledast G$, for some r; in this example $\widetilde{H}$ itself happens to be in $k_t \circledast G$, so we could construct a smaller graph. Following the construction in the proof, since $m = k_1 = k_2 = k_3 = k_4 = 2$, the final graph will contain $m^4 k_1 k_2 k_3 k_4 + 2 = 258$ copies of G, so the reader can appreciate why we did not attempt to draw this graph. Even with the shortcut indicated above, we would get a graph with 18 copies of G.

Is a $\mathcal{P}$-decomposition of G with few parts simply a coarser version of a decomposition with more parts? If $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_r$, is a $\mathcal{P}$-decomposition also a $(\mathcal{P}_1, \ldots, \mathcal{P}_r)$-partition? We know, by Lemma 4.1.2, that a uniquely $\mathcal{P}$-decomposable graph is uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_r)$-partitionable, but is the converse true? We can now give some answers to these questions (see also Theorem 5.3.2(3)).

4.2.3. Corollary. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a $\mathcal{P}$-decomposition of G with exactly $dec(\mathcal{P})$ parts that respects d_0.*

Proof: In Theorem 4.2.2, since $G^* \geq G$ we know G^* is $\mathcal{P}$-strict, and so $dec(\mathcal{P}) \leq dec_{\mathcal{P}}(G^*) \leq dec_{\mathcal{P}}(G) = dec(\mathcal{P})$. Thus G^* has at least one $\mathcal{P}$-decomposition d with $dec(\mathcal{P})$ parts; $d|G$ also has $dec(\mathcal{P})$ parts (since G is $\mathcal{P}$-strict) and respects d_0. $\square$

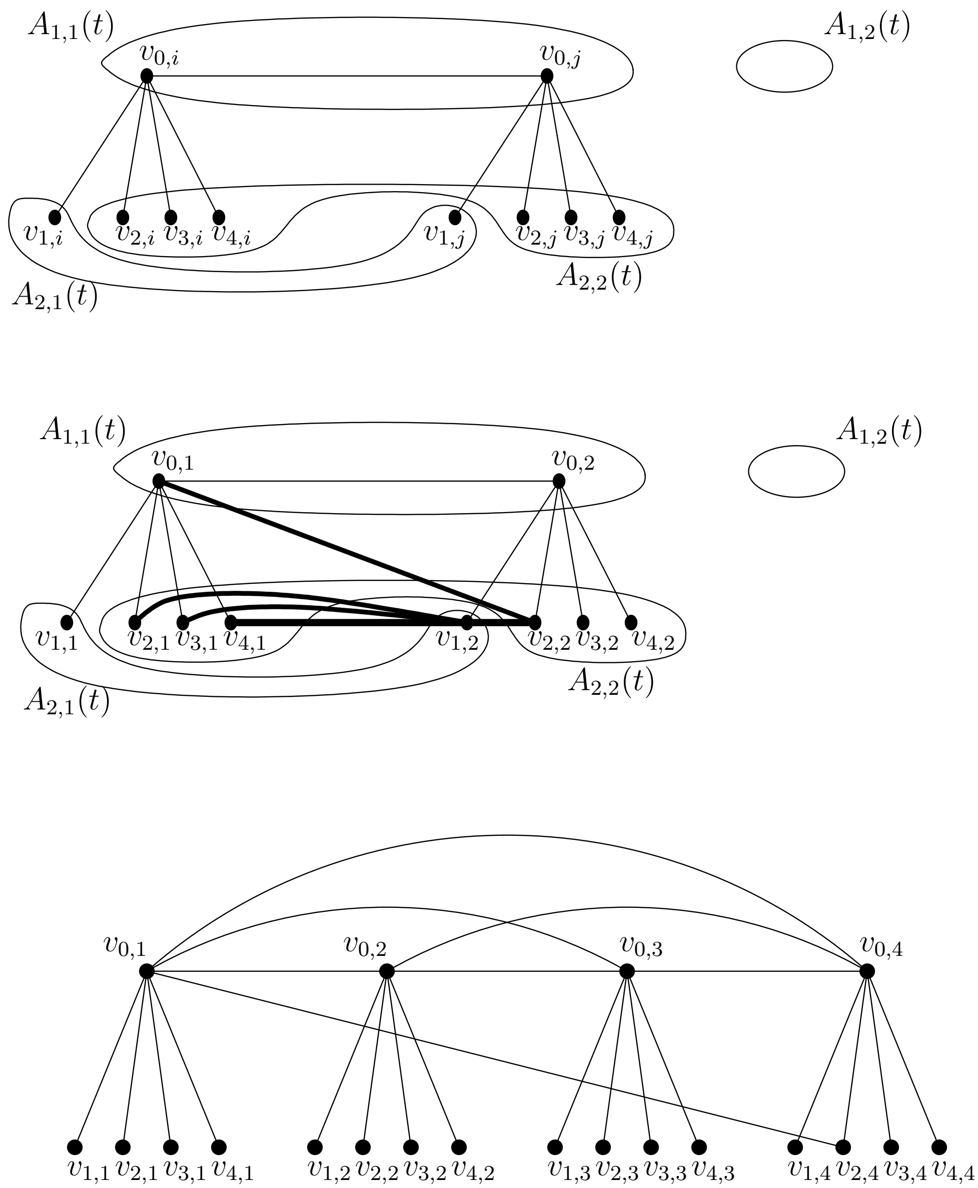


Figure 4.9: G_{k_1}, $\widetilde{H}$ and $m.k_1 \bullet G$

4.2.4. Corollary [52]. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = n$, and let $d_0 = (U_1, U_2, \ldots, U_n)$ be a fixed $\mathcal{P}$-decomposition of G with n parts. Then there is a $\mathcal{P}$-strict graph $G^* \in s\circledast G$ (for some s) which has a unique $\mathcal{P}$-decomposition d with n parts, and $d|G^j = d_0$ for all j.*

Proof: The only $\mathcal{P}$-decomposition of G with n parts that respects d_0 is d_0 itself (since here d_0 has exactly n parts). Thus in Theorem 4.2.2, the only possible decomposition of G^* with n parts is the extension of d_0, which is a $\mathcal{P}$-decomposition of G^* by construction. $\square$

The next result tells us that under certain conditions, given a factorisation $\mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$ of $\mathcal{P}$ into indiscompositive properties, and a $\mathcal{P}$-decomposition d_0 of G, we can group the parts of d_0 to get a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G. Of course, d_0 does not respect *all* $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partitions; in fact, if $m = dec(\mathcal{P})$, d_0 can only respect one partition, namely d_0 itself (note that none of the parts of a partition can be empty, because G is $\mathcal{P}$-strict). We will actually use Corollary 4.2.5 to show (Theorem 4.3.3) that when we factor the $\mathcal{Q}_i$'s as far as possible we do get exactly $dec(\mathcal{P})$ irreducible factors, say $\mathcal{P}_1, \ldots, \mathcal{P}_{dec(\mathcal{P})}$, and applying the corollary we get that d_0 *is* a $(\mathcal{P}_1, \ldots, \mathcal{P}_{dec(\mathcal{P})})$-partition.

4.2.5. Corollary. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$. Then for any $\mathcal{P}$-decomposition $d_0 = (U_1, U_2, \ldots, U_{dec(\mathcal{P})})$ of G, there is a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G that d_0 respects.*

Proof: The graph G^* of Corollary 4.2.4 has some $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition d_1; this is also a $\mathcal{P}$-decomposition. Now G^* has a unique $\mathcal{P}$-decomposition d with $dec(\mathcal{P})$ parts; by Corollary 4.2.3 d must respect d_1; and the restriction of d to G is just d_0. $\square$

Recall that $\mathbf{S} := \mathbf{S}(\mathcal{P})$ is the set of $\mathcal{P}$-strict graphs, and $\mathbf{S}^{\downarrow}$ the set of $\mathcal{P}$-strict graphs with decomposability $dec(\mathcal{P})$. The set of $\mathcal{P}$-strict, uniquely $\mathcal{P}$-decomposable graphs with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$ is denoted $\mathbf{S}^{\Downarrow}(\mathcal{P})$, or just $\mathbf{S}^{\Downarrow}$. So $\mathbf{S}^{\Downarrow} \subseteq \mathbf{S}^{\downarrow} \subseteq \mathbf{S}$. We already know that $\mathbf{S}$ generates $\mathcal{P}$; so does $\mathbf{S}^{\downarrow}$ (by Lemma 4.1.7) and even $\mathbf{S}^{\Downarrow}$ (by Corollary 4.2.4). In fact, for any $G \in \mathbf{S}^{\downarrow}$ and any $\mathcal{P}$-decomposition d of G, we can find an induced supergraph in $\mathbf{S}^{\Downarrow}$ whose ind-parts uniformly respect d.

4.2.6. Corollary. *Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$, and let $d_0 = (U_1, U_2, \ldots, U_m)$ be a fixed $\mathcal{P}$-decomposition of G. Then there is a uniquely $\mathcal{P}$-decomposable $\mathcal{P}$-strict graph $G^* \geq G$ whose ind-parts respect d_0 uniformly.* □

4.2.7. Corollary. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{dec(\mathcal{P})}$. Let G be a $\mathcal{P}$-strict graph with $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$. If $d_0 = (U_1, U_2, \ldots, U_m)$ is a $\mathcal{P}$-decomposition of G, then there is a factorisation $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$ such that d_0 is a $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G.*

Proof: By Corollary 4.2.6 there is a a uniquely $\mathcal{P}$-decomposable graph $G^* \geq G$ whose ind-parts respect d_0 uniformly. So if $(V_1, V_2, \ldots, V_{dec(\mathcal{P})})$ is the unique $\mathcal{P}$-decomposition of G^*, then there is a partition $(J_1, J_2, \ldots, J_m)$ of $\{1, 2, \ldots, dec(\mathcal{P})\}$ such that, for each i, $U_i = \cup_{j \in J_i} V_j$ (when we restrict the V_j to a particular copy of G in G^*).

By Lemma 4.1.2, the ind-parts of G^* form its unique $(\mathcal{P}_1, \ldots, \mathcal{P}_{dec(\mathcal{P})})$-partition; therefore $G[U_i] \in \prod_{j \in J_i} \mathcal{P}_j$, so we may set $\mathcal{Q}_i = \prod_{j \in J_i} \mathcal{P}_j$. □

4.2.8. Proposition. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$, where the $\mathcal{P}_i$'s are indiscompositive. Let $G \in \mathbf{S}^{\downarrow}(\mathcal{P})$ be in $G_1 * \cdots * G_m$, with each $G_i \in \mathbf{S}^{\downarrow}(\mathcal{P}_i)$. Then there is a uniquely $\mathcal{P}$-decomposable graph $L \in s \circledast G$ (for some s), with ind-parts $L_1, \ldots, L_{dec(\mathcal{P})}$, and a partition $(J_1, \ldots, J_m)$ of $\{1, 2, \ldots, n\}$ such that, for each $1 \leq i \leq m$, $\cup_{j \in J_i} V(L_j)$ induces a graph $G_i^s \in s \circledast G_i$ that is in $\mathbf{S}^{\downarrow}(\mathcal{P}_i)$.*

Proof: Let $d_1 = (V_1, \ldots, V_m)$ be a $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition of G, where $G_i = G[V_i]$. Since $\mathcal{P}_i$ is disjoint compositive, there is a tournament sequence for G_i, with, say, $G_i^r \in r \circledast G_i$ in $\mathcal{P}_i$. We can add edges between $G_1^r, \ldots, G_m^r$ to get a graph $G^r \in r \circledast G$, so that $G^1, G^2, G^3, \ldots$ is a tournament sequence for G (see Figure 4.10). By Theorem 4.2.2 and Corollary 4.2.6 we can use this tournament sequence to construct a graph $L \in s \circledast G$ (for some s) in $\mathbf{S}^{\Downarrow}(\mathcal{P})$ whose ind-parts respect d_1 uniformly. By construction, L is obtained from the G^s without changing any edges *within* the m parts.

Thus, denoting the ind-parts of L by $L_1, \ldots, L_n$, there is a partition $(J_1, J_2, \ldots, J_m)$ of $\{1, 2, \ldots, n\}$ such that, for each i, $\cup_{j \in J_i} V(L_j)$ induces the graph $G_i^s \in s \circledast G_i$ from the tournament sequence in $\mathcal{P}_i$. By Lemma 4.1.4, G_i^s is $\mathcal{P}_i$-strict with $dec_{\mathcal{P}_i}(G_i^s) = dec(\mathcal{P}_i)$. □

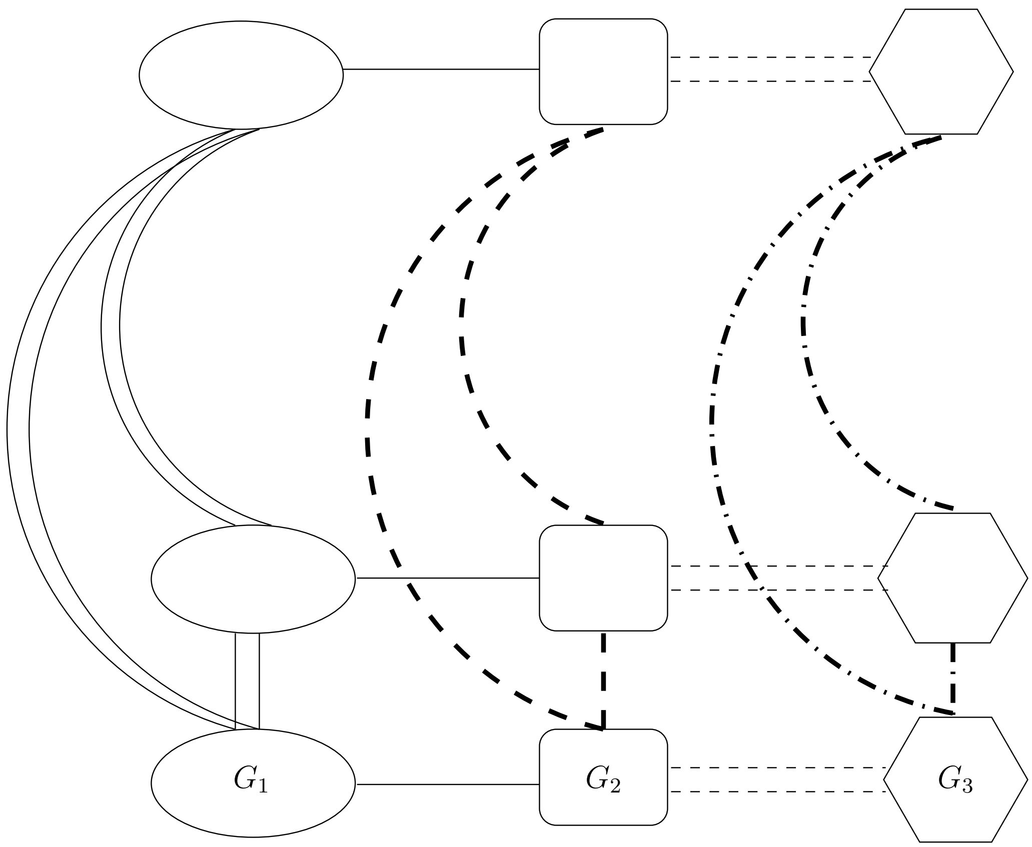


Figure 4.10: G^r

4.3 Unique factorisation for indiscompositive properties

The strategy for proving the uniqueness of the factorisation for indiscompositive properties is the same as that used in Section 3.3 for hereditary compositive properties. We shall first show that there is at most one factorisation with $dec(\mathcal{P})$ factors and then that any such factorisation must have $dec(\mathcal{P})$ factors.

An indiscompositive property $\mathcal{P}$ is *indecomposable* if $dec(\mathcal{P}) = 1$. An indecomposable property is irreducible over $\mathbb{L}^{dc}_{\leq}$, and Mihók [52] showed that the converse is also true (see Theorem 4.3.5).

The following construction of a generating set for $\mathcal{P}$ will be essential in proving unique factorisation. Suppose we are given a factorisation $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ into indecomposable indiscompositive factors, and, for each i, we are given a generating set $\mathcal{G}_i$ of $\mathcal{P}_i$ and a graph $H_i \in \mathcal{P}_i$. By Lemmas 4.1.6 and 4.1.7, the set $\mathcal{G}_i^{\downarrow}[H_i] := \{G \in (\mathcal{G}_i \cap \mathbf{S}(\mathcal{P}_i)) \mid H_i \leq G,\ dec_{\mathcal{P}_i}(G) = 1\}$ is also a generating set for $\mathcal{P}_i$.

The $*$-join of these m sets is then a generating set for $\mathcal{P}$, and we can once again pick out just those graphs that are strict and have minimum decomposability:

$$(\mathcal{G}_1[H_1] * \cdots * \mathcal{G}_m[H_m])^{\downarrow} := \{G' \in \mathbf{S}(\mathcal{P}) \mid dec_{\mathcal{P}}(G') = dec(\mathcal{P}),\ \text{and } \forall\, i,\\ 1 \leq i \leq m,\ \exists\, G_i \in \mathcal{G}_i^{\downarrow}[H_i],\ G' \in G_1 * \cdots * G_m\}.$$

4.3.1. Lemma. *Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$. Then: $\mathcal{G} = (\mathcal{G}_1[H_1] * \cdots * \mathcal{G}_m[H_m])^{\downarrow} \subseteq \mathbf{S}(\mathcal{P})$ is a generating set for $\mathcal{P}$; every $G \in \mathcal{G}$ has $dec_{\mathcal{P}}(G) = dec(\mathcal{P})$; and, $\forall G \in \mathcal{G}$, $\forall i = 1, \ldots, m$, $\exists G_i$ that is $\mathcal{P}_i$-indecomposable, $H_i \leq G_i$ and $G \in G_1 * \cdots * G_m$.* □

We are now ready to prove unique factorisation. As in the hereditary case, we first show that any two factorisations with exactly $dec(\mathcal{P})$ indecomposable factors must be the same, and then prove that any factorisation into indecomposable factors must have exactly $dec(\mathcal{P})$ terms.

4.3.2. Theorem. *An indiscompositive property $\mathcal{P}$ can have only one factorisation with exactly $dec(\mathcal{P})$ indecomposable factors.*

Proof: Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_n$ be two factorisations of $\mathcal{P}$ into $n = dec(\mathcal{P})$ indecomposable factors. Label the $\mathcal{P}_i$'s inductively, beginning with $i = n$, so that, for each i, $\mathcal{P}_i$ is inclusion-wise maximal among $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_i$. For each i, j such that $i > j$, if $\mathcal{P}_i \setminus \mathcal{P}_j \neq \emptyset$, then let $X_{i,j} \in \mathcal{P}_i \setminus \mathcal{P}_j$; if $\mathcal{P}_i \setminus \mathcal{P}_j = \emptyset$, then $\mathcal{P}_i = \mathcal{P}_j$ and we set $X_{i,j}$ to be the null graph. For each i, by indiscompositivity there is an $H_{i,0} \in \mathcal{P}_i$ that contains all the $X_{i,j}$'s as induced-subgraphs. The important point is that if $\{L_1, L_2, \ldots, L_n\}$ is an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of some graph G such that, for each $i = 1, 2, \ldots, n$, $H_{i,0} \leq G[L_i]$, then, by reverse induction on i starting at n, $G[L_i] \in \mathcal{P}_i$.

For each i, let $\mathcal{G}_i = \{G_{i,0}, G_{i,1}, G_{i,2}, \ldots\}$ be a generating set for $\mathcal{P}_i$. We will construct another generating set for each $\mathcal{P}_i$ that will turn out to be contained in some $\mathcal{Q}_j$; for graphs $G_{i,s}, H_{i,s}$, we will use the second subscript to denote which step of our construction we are in.

For each $s \geq 0$, choose a graph $H'_{s+1} \in (\mathcal{G}_1[H_{1,s}, G_{1,s}]*\cdots*\mathcal{G}_n[H_{n,s}, G_{n,s}])^{\downarrow}$, and find an induced supergraph H_{s+1} whose unique $\mathcal{P}$-decomposition with $dec(\mathcal{P})$ parts uniformly respects the obvious decomposition of H'_{s+1}. We label as $H_{i,s+1}$ the ind-part of H_{s+1} that contains the graph from $\mathcal{G}_i[H_{i,s}, G_{i,s}]$. Then, for each i, $H_{i,0} \leq H_{i,1} \leq H_{i,2} \leq \cdots$

For $\mathcal{G}_i[H_{i,s}, G_{i,s}]$ to be non-empty, we must have $H_{i,s} \in \mathcal{P}_i$. We know that the $H_{i,s+1}$'s give an unordered $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition of H_{s+1}. From the earlier remark, for $i = 1, 2, \ldots, n$, $H_{i,s+1} \in \mathcal{P}_i$.

The ind-parts of H_s also form its unique unordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$-partition. Thus, there is some permutation φ_s of $\{1, 2, \ldots, n\}$ such that, for each i, $H_{i,s} \in \mathcal{Q}_{\varphi_s(i)}$. Since there are only finitely many permutations of $\{1, 2, \ldots, n\}$, there must be some permutation φ that appears infinitely often. Now, whenever $\varphi_t = \varphi$, we have $H_{i,1} \leq H_{i,2} \leq \cdots \leq H_{i,t} \in \mathcal{Q}_{\varphi(i)}$ so by induced-heredity, for every $s \leq t$, $H_{i,s}$ is in $\mathcal{Q}_{\varphi(i)}$. Therefore, we can take $\varphi_s = \varphi$, for all s. By re-labelling the $\mathcal{Q}_i$'s, we can assume φ is the identity permutation, so that $H_{i,s} \in \mathcal{Q}_i$ for all i and s.

Now for each i and s, $G_{i,s-1} \leq H_{i,s}$, so that $\mathcal{H}_i := \{H_{i,1}, H_{i,2}, \ldots\}$ is a generating set for $\mathcal{P}_i$. But $\mathcal{H}_i \subseteq \mathcal{Q}_i$, so $\mathcal{P}_i = (\mathcal{H}_i)_{\leq} \subseteq \mathcal{Q}_i$.

By the same reasoning, there is a permutation τ such that $\mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)}$. We cannot relabel the $\mathcal{P}_i$'s as well, but if $\tau^k(i) = i$, then we have $\mathcal{P}_i \subseteq \mathcal{Q}_i \subseteq \mathcal{P}_{\tau(i)} \subseteq \mathcal{Q}_{\tau(i)} \subseteq \mathcal{P}_{\tau^2(i)} \subseteq \mathcal{Q}_{\tau^2(i)} \subseteq \cdots \subseteq \mathcal{P}_{\tau^k(i)} = \mathcal{P}_i$, so we must have equality throughout; in particular, $\mathcal{P}_i = \mathcal{Q}_i$ for each i. □

The second piece is analogous to Theorem 3.3.1, but the technical details are rather different.

4.3.3. Theorem. *Let $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ be a factorisation of the indiscompositive property $\mathcal{P}$ into indecomposable indiscompositive properties. Then $m = dec(\mathcal{P})$.*

Proof: By Lemma 4.1.2 any $\mathcal{P}$-strict graph G has $dec_{\mathcal{P}}(G) \geq m$, so $dec(\mathcal{P}) \geq m$. To prove the reverse inequality, we suppose $m < n := dec(\mathcal{P})$ and then construct a sequence of graphs until we get a contradiction. When graphs or sets have a double subscript, we will use the second number to denote which step of our construction we are in.

For each i, we start with some generating set $\mathcal{G}_i$ consisting only of $\mathcal{P}_i$-

indecomposable $\mathcal{P}_i$-strict graphs. Let G_1 be in $(\mathcal{G}_1 * \cdots * \mathcal{G}_m)^{\downarrow}$, with $G_1 \in G_{1,1} * \cdots * G_{m,1}$, where each $G_{i,1}$ is $\mathcal{P}_i$-strict and $\mathcal{P}_i$-indecomposable. By Proposition 4.2.8 there is a graph H_1 satisfying conditions (a)–(d) below.

In general, suppose we have graphs $H_1, H_2, \ldots, H_{\ell-1}$ such that, for each $k = 1, 2, \ldots, \ell - 1$:

(a) H_k is $\mathcal{P}$-strict and uniquely $\mathcal{P}$-decomposable;

(b) $dec_{\mathcal{P}}(H_k) = n$, with ind-parts $H_{1,k}, \ldots, H_{n,k}$;

(c) $H_1 \leq H_2 \leq \cdots$, with the ind-parts labelled such that, for $j = 1, \ldots, n$, $H_{j,1} \leq H_{j,2} \leq \cdots$;

(d) there is a partition $(J_{1,k}, J_{2,k}, \ldots, J_{m,k})$ of $\{1, 2, \ldots, n\}$ such that, for each i, the union $\cup_{j \in J_{i,k}} V(H_{j,k})$ induces a $\mathcal{P}_i$-indecomposable graph $G'_{i,k}$; and

(e) there is an $i = i(k)$ such that, for $p > k$, $\cup_{j \in J_{i,k}} V(H_{j,p})$ does not induce a graph in $\mathcal{P}_i$ (note the change in subscript).

For $\ell > 1$, we will construct H_ℓ to satisfy (a)–(d), and condition (e), when $k = \ell - 1$ and $p = \ell$; by (c), condition (e) will then hold for *all* $p > \ell - 1$.

We will find graphs H'_ℓ and H''_ℓ before constructing H_ℓ itself (see Figures 4.11, 4.12). Relative to the $\mathcal{P}$-decomposition given by the $H_{j,(\ell-1)}$'s, there is some tournament sequence $H^1_{\ell-1} \leq H^2_{\ell-1} \leq \cdots$, where $H^r_{\ell-1} \in r \circledast H_{\ell-1}$. Let $H^r_{j,(\ell-1)}$ be the subgraph of $H^r_{\ell-1}$ that is in $r \circledast H_{j,(\ell-1)}$.

Because $m < n$, there is an $i := i(\ell - 1)$ such that $G'_{i,(\ell-1)}$ contains more than one ind-part. But $G'_{i,(\ell-1)}$ is $\mathcal{P}_i$-indecomposable, so, for large enough r, there is some $H'_\ell \in H^r_{1,(\ell-1)} * \cdots * H^r_{n,(\ell-1)}$ for which $H'_\ell[\cup_{j \in J_{i,(\ell-1)}} V(H^r_{j,(\ell-1)})]$ is not in $\mathcal{P}_i$. We can also require that $H_{\ell-1} \leq H'_\ell$ (by taking H'_ℓ from $H^{r+1}_{\ell-1}$ instead of $H^r_{\ell-1}$). By Corollary 4.2.4, we can take r even larger (much larger) to get an H'_ℓ that is uniquely $\mathcal{P}$-decomposable; then $H'_{j,\ell} := H^r_{j,(\ell-1)}$ is an ind-part of H'_ℓ, labeled such that $H_{j,(\ell-1)} \leq H'_{j,\ell}$.

H'_ℓ is still in $\mathcal{P}$, and is $\mathcal{P}$-strict with $\mathcal{P}$-decomposability $dec(\mathcal{P})$ (because it contains $H_{\ell-1}$). By Corollary 4.2.5, it has some $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $(W'_1, \ldots, W'_m)$ such that each W'_i is a union of one or more $H'_{j,\ell}$.

H'_ℓ contains a copy of $H_{\ell-1}$, but its copy of $G_{i,(\ell-1)}$ need not be in $H'_\ell[W'_i]$. However, by disjoint compositivity of the $\mathcal{P}_i$'s, there is a graph $H''_\ell \in \mathcal{P}$ containing H'_ℓ and $H_{\ell-1}$ as disjoint induced-subgraphs, with some $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition $d_\ell = (W_1, \ldots, W_m)$ such that $H''_\ell[W_i]$ contains $H'_\ell[W'_i]$ and $G'_{i,(\ell-1)}$

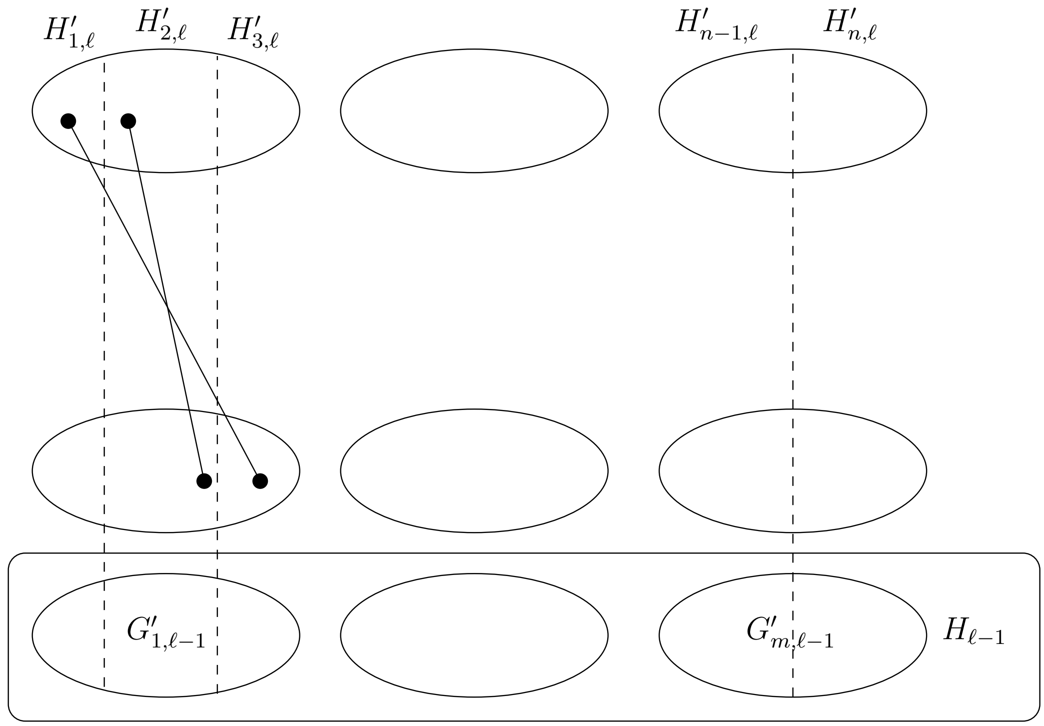


Figure 4.11: H'_ℓ contains $H_{\ell-1}$, but the extension of $G'_{1,(\ell-1)}$ is not in $\mathcal{P}_1$.

as disjoint induced-subgraphs. By Lemma 4.1.4, $H''_\ell[W_i]$ is $\mathcal{P}_i$-strict and $\mathcal{P}_i$-indecomposable, while H''_ℓ is $\mathcal{P}$-strict with $\mathcal{P}$-decomposability n. Using Proposition 4.2.8 again, there is a graph $H_\ell \geq H''_\ell$ in $\mathbf{S}^{\Downarrow}(\mathcal{P})$ whose ind-parts uniformly respect d_ℓ.

Properties (a) and (b) hold for H_ℓ by virtue of being in $\mathbf{S}^{\Downarrow}$. Since $H_{\ell-1} \leq H'_\ell \leq H_\ell$, and $H_{\ell-1}$ is uniquely $\mathcal{P}$-decomposable, by Lemma 4.2.1 we can label the ind-parts of H_ℓ as $H_{1,\ell}, \ldots, H_{n,\ell}$ such that $H_{j,(\ell-1)} \leq H_{j,\ell}$, thus satisfying (c). Condition (d) follows from our use of Proposition 4.2.8. Condition (e) holds when $k = \ell - 1$ and $p = \ell$, because of the "bad" subgraph H'_ℓ of H_ℓ; as noted previously, condition (c) then guarantees that (e) holds for all $p > \ell - 1$.

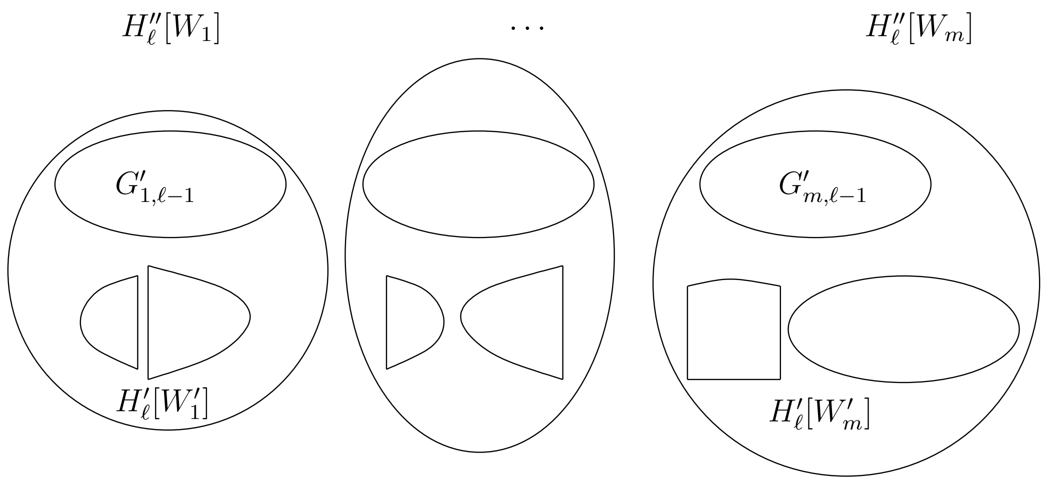


Figure 4.12: Constructing H''_ℓ from H'_ℓ and $H_{\ell-1}$.

Since there is only a finite number of partitions of $\{1, 2, \ldots, n\}$, at some step B we must end up with a partition that occurred at a previous step $A < B$. But then (d) contradicts (e). $\square$

The following result has a 2-page proof along the lines of that for Theorem 3.2.2 (though it now becomes technically more difficult to show that $dec(\mathcal{Q}_F) \geq k$ and $dec(\mathcal{Q}_{\overline{F}}) \geq n-k$ — see the exposition in [6]). We present here a simpler proof that is due to Jim Geelen.

4.3.4. Theorem [52]**.** *An indiscompositive property $\mathcal{P}$ has a factorisation $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{dec(\mathcal{P})}$ into $dec(\mathcal{P})$ (necessarily indecomposable) indiscompositive factors. Moreover, when $\mathcal{P}$ is additive, the factors can be taken to be additive too.*

Proof: By Corollary 4.2.4, Theorem 2.3.4 and Lemma 4.1.7, $\mathcal{P}$ has an ordered generating set $H_1 < H_2 < \cdots$, where each H_i is uniquely $\mathcal{P}$-decomposable with decomposability $dec(\mathcal{P})$. By Lemma 4.2.1 we can label the ind-parts as $H_{1,i}, \ldots, H_{d,i}$, so that, for $j = 1, \ldots, d$, $H_{j,1} \leq H_{j,2} \leq H_{j,3} \leq \cdots$. Let $\mathcal{P}_j$ be the induced-hereditary property generated by the $H_{j,i}$'s. We claim that $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_d$.

If G is in $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_d$, then $V(G)$ has a partition $\{V_1, \ldots, V_d\}$ such that, for each j, $G[V_j] \in \mathcal{P}_j$. So there exist $k_1, \ldots, k_d$ such that, for each j, $G[V_j] \subseteq H_{j,k_j}$. Taking $k := \max\{k_1, \ldots, k_d\}$ we have $G \in H_{1,k} * \cdots * H_{d,k} \subseteq \mathcal{P}$. Conversely, if G is in $\mathcal{P}$, it is contained in some H_k, and it is easy to see that it has a $(\mathcal{P}_1, \ldots, \mathcal{P}_d)$-partition.

We need to show that, if $\mathcal{P}$ is additive, the $\mathcal{P}_j$'s are also additive (the argument for indiscompositive properties is identical). For each $1 \leq r, s \leq d$ such that $\mathcal{P}_r \setminus \mathcal{P}_s \neq \emptyset$, fix a graph $X_{r,s} \in (\mathcal{P}_r \setminus \mathcal{P}_s)$. By omitting some graphs from our ordered generating set, we can assume that $X_{r,s} \leq H_{r,1}$ for each r and s.

To prove additivity of $\mathcal{P}_j$ it is sufficient to show that, for all i, $2H_{j,i}$ is contained in some $H_{j,i'}$. By additivity of $\mathcal{P}$, $(d!+1)H_i$ is contained in some $H_{i'}$. By Lemma 4.2.1, for each copy of H_i, there is a permutation ϕ of $\{1, \ldots, d\}$ such that, for each k, $H_{k,i} \leq H_{\phi(k),i'}$. Since there are only $d!$ possible permutations, there are two copies of H_i with the same permutation, so for some ϕ we actually have, for each k, $2H_{k,i} \leq H_{\phi(k),i'}$.

Now, $\mathcal{P}_j \subseteq \mathcal{P}_{\phi(j)}$ (otherwise $X_{j,\phi(j)} \notin \mathcal{P}_{\phi(j)}$ is contained in $H_{j,i} \subseteq H_{\phi(j),i'}$, a contradiction). If $\phi^t(j) = j$, then by repeating this argument, we get $\mathcal{P}_j \subseteq \mathcal{P}_{\phi(j)} \subseteq \mathcal{P}_{\phi^2(j)} \subseteq \cdots \subseteq \mathcal{P}_{\phi^t(j)} = \mathcal{P}_j$. So $2H_i^j \leq H_{i'}^{\phi(j)} \in \mathcal{P}_{\phi(j)} = \mathcal{P}_j$. $\square$

4.3.5. Corollary [52]. *An indiscompositive property is irreducible over $\mathbb{L}_{\leq}^{dc}$ if and only if it is indecomposable. An additive induced-hereditary property is irreducible over $\mathbb{L}_{\leq}^{a}$ iff it is irreducible over $\mathbb{L}_{\leq}^{dc}$ iff it is indecomposable.* $\square$

4.3.6. Corollary. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be irreducible indiscompositive properties. Then there is a uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graph G.*

Proof: Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. By Corollary 4.3.5 the $\mathcal{P}_i$'s are indecomposable, and by Theorem 4.3.3, $dec(\mathcal{P}) = n$. By Corollary 4.2.4 there is a graph in $\mathcal{P}$ with a unique $\mathcal{P}$-decomposition with exactly n parts, and by Lemma 4.1.2 it must be uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable. $\square$

4.3.7. Induced-hereditary Unique Factorisation Theorems. *An indiscompositive property has a unique factorisation into properties that are irreducible over $\mathbb{L}_{\leq}^{dc}$; an additive induced-hereditary property has a unique factorisation into properties that are irreducible over $\mathbb{L}_{\leq}^{a}$. In each case, the number of factors is exactly $dec(\mathcal{P})$.* $\square$

Chapter 5

Reducibility, co-primality, and uniquely colourable graphs

We now have unique factorisation theorems for four classes of properties ($\mathbb{P} \in \{\mathbb{L}^c, \mathbb{L}^a, \mathbb{L}^a_\leq, \mathbb{L}^{dc}_\leq\}$), of the form:

> *Every property $\mathcal{P} \in \mathbb{P}$ is uniquely factorisable into properties that are irreducible over $\mathbb{P}$.*

Do the properties in $\mathbb{P}$ remain uniquely factorisable when we allow factors that are outside $\mathbb{P}$? In particular, are properties irreducible over $\mathbb{P}$ actually irreducible (over $\mathbb{U}$)? In this chapter we show that the answers are No (or not always) and Yes.

First, in Section 5.1, we give some direct consequences of unique factorisation, and make some related observations; in particular, properties in $\mathbb{L}^a$ are uniquely factorisable over $\mathbb{L}^c$ or $\mathbb{L}^{dc}_\leq$. Then in Section 5.2 we show that, with a few well-described exceptions, a property that factors into at least two induced-hereditary properties also has a factorisation with at least one non-induced-hereditary factor; this is the first main result of this chapter.

Section 5.3 centers on uniquely $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-colourable graphs, and particularly Broere and Bucko's characterisation of the sets of $\mathcal{Q}_i$'s for which such graphs exist. We use these to show how a simple-minded approach will extract generating sets for $\mathcal{A}$ and $\mathcal{B}$ from a generating set for $\mathcal{A} \circ \mathcal{B}$. It is an easy consequence of Broere and Bucko's characterisation that, for the classes above, the irreducible properties in $\mathbb{P}$ are precisely the properties irreducible over $\mathbb{P}$; this is the other main result of the chapter, stated in Section 5.4.

5.1 Consequences and parallels of unique factorisation

Before proving the uniqueness of the factorisations in [51] and [52], we tried without success to prove some related results. Their validity for general induced-hereditary properties is mostly still open. However, for indiscompositive and hereditary compositive properties these results follow quite easily from Unique Factorisation, and we state them explicitly in this section. We also consider some easy related unique factorisation results.

We start by noting that $\mathbb{L}^c \cap \mathbb{L}^{dc}_{\leq} = \mathbb{L}^a$, and for properties in $\mathbb{L}^a$ it now follows that the factorisation of [51] is the same as that of [52].

5.1.1. Proposition. *Let $\mathcal{P}$ be an additive hereditary property. Then $dc(\mathcal{P}) = dec(\mathcal{P})$. If $\mathcal{P} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_r$, where the $\mathcal{Q}_j$'s are all in $\mathbb{L}^c$, or all in $\mathbb{L}^{dc}_{\leq}$, then the $\mathcal{Q}_j$'s are in $\mathbb{L}^a$.*

Proof: By Theorem 3.2.2, $\mathcal{P}$ has a factorisation $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_{dc(\mathcal{P})}$ into $dc(\mathcal{P})$ additive hereditary factors, which by Lemma 4.1.2 implies $dc(\mathcal{P}) \leq dec(\mathcal{P})$. By Theorem 4.3.4, $\mathcal{P}$ also has a factorisation $\mathcal{R}_1 \circ \cdots \circ \mathcal{R}_{dec(\mathcal{P})}$ into $dec(\mathcal{P})$ additive induced-hereditary properties. Now, using Proposition 2.1.3, $\mathcal{P} = \mathcal{R}_1 \circ \cdots \circ \mathcal{R}_{dec(\mathcal{P})} \subseteq (\mathcal{R}_1)_\subseteq \circ \cdots \circ (\mathcal{R}_{dec(\mathcal{P})})_\subseteq = (\mathcal{R}_1 \circ \cdots \circ \mathcal{R}_{dec(\mathcal{P})})_\subseteq = \mathcal{P}_\subseteq = \mathcal{P}$, so we have equality throughout. Since $\mathcal{R}_i$ is additive, $(\mathcal{R}_i)_\subseteq$ is additive, so there is a factorisation of $\mathcal{P}$ into $dec(\mathcal{P})$ additive hereditary properties, and thus $dec(\mathcal{P}) \leq dc(\mathcal{P})$, by Lemma 3.2.1. In fact, by unique factorisation, the $\mathcal{R}_i$'s must themselves be additive *hereditary*, so they are just the $\mathcal{P}_i$'s.

Now, if the $\mathcal{Q}_j$'s are all hereditary compositive, and we factor them into properties that are irreducible over $\mathbb{L}^c$, then by Theorem 3.3.3 (cf. Corollary 3.2.3) we must get the factorisation into $\mathcal{P}_i$'s. Similarly, if the $\mathcal{Q}_j$'s are all indiscompositive, and we factor them into properties that are irreducible over $\mathbb{L}^{dc}_{\leq}$, then by Theorem 4.3.7 (cf. Corollary 4.3.5) we must get the factorisation into $\mathcal{P}_i$'s. In either case, each $\mathcal{Q}_j$ is the product of additive hereditary factors, so it is itself additive hereditary. $\square$

Since a property that is not indiscompositive has decomposability 0, there is little point in discussing how far we can extend the first part of Proposition 5.1.1 or the second part of Proposition 5.1.2. However, the first part of Proposition 5.1.2 holds even for the hereditary properties given by Mihók et al. (see p. 42) that are a counterexample to unique factorisation — $dc(\mathcal{P}_1)+dc(\mathcal{P}_2) = dc(\mathcal{P}_1 \circ \mathcal{P}_2) = dc(\mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \mathcal{Q}_3) = dc(\mathcal{Q}_1)+dc(\mathcal{Q}_2)+dc(\mathcal{Q}_3)$.

Proposition 5.1.2 also implies Theorems 3.3.1 and 4.3.5, but not Theorems 3.2.3 or 4.3.5 (which explains how it can hold for the properties above).

5.1.2. Proposition. *If $\mathcal{P}$ and $\mathcal{Q}$ are hereditary compositive properties, then $dc(\mathcal{P} \circ \mathcal{Q}) = dc(\mathcal{P}) + dc(\mathcal{Q})$. If $\mathcal{P}$ and $\mathcal{Q}$ are indiscompositive properties, then $dec(\mathcal{P} \circ \mathcal{Q}) = dec(\mathcal{P}) + dec(\mathcal{Q})$.* □

5.1.3. Cancellation Theorem. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ all be indiscompositive properties (or all hereditary compositive). If $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \circ \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.* □

5.1.4. Corollary. *Let $\mathcal{A}', \mathcal{A}, \mathcal{B}', \mathcal{B}$ all be indiscompositive properties (or all hereditary compositive). If $\mathcal{A}' \circ \mathcal{B}' = \mathcal{A} \circ \mathcal{B}$, and $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{B}' \subseteq \mathcal{B}$, then $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}' = \mathcal{B}$.* □

One might wonder whether such intuitive results can be established without going to the effort of proving unique factorisation. It is therefore instructive to see that the results need not hold even in settings just as natural as that of hereditary properties (see also the discussion before Theorem 6.1.2).

A property $\mathcal{P}$ is $\supseteq$-*hereditary* if all supergraphs of a graph in $\mathcal{P}$ are also in $\mathcal{P}$, i.e., $(G \supseteq H$ and $H \in \mathcal{P}) \Rightarrow G \in \mathcal{P}$. Such a property is automatically infinite and additive, and the class $\mathbb{L}_{\supseteq}$ of all such properties is closed under products. The $\supseteq$-hereditary properties are exactly the setwise complements (with respect to the set of all non-null finite graphs) of $\subseteq$-hereditary properties, and they display almost opposite behaviour. If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$, and this is true also for $\geq$-hereditary properties, and for coverings of $\mathcal{P}$ by three or more sets. We are using the convention that empty parts are allowed in a $(\mathcal{P}_1, \mathcal{P}_2)$-partition. When no empty parts are allowed, it is easy to see [20] that if $\mathcal{P}$ is the product of r factors, then $r \leq \min\{|V(G)| \mid G \in \mathcal{P}\}$. We summarise these remarks in the following.

5.1.5. Lemma. *For every integer $r > 1$, any property in $\mathbb{L}_{\supseteq}$ or $\mathbb{L}_{\geq}$ has uncountably many factorisations into r properties from $\mathbb{L}_{\supseteq}$ or $\mathbb{L}_{\geq}$.* □

In particular, no $\geq$-hereditary property is irreducible, even over $\mathbb{L}_{\geq}$. And yet we can specify a canonical factorisation quite easily.

For any graph G, let $-G := \{H \mid G \not\leq H\}$ and $+G := \{H \mid G \leq H\}$. Properties of the form $-G$ are *elementary* (cf. [22]) because every induced-hereditary property $\mathcal{P}$ can be expressed as $\mathcal{P} = \bigcap\{-G \mid G \in \mathcal{F}_{\leq}(P)\}$ (compare also Theorem 2.2.2). By Theorem 2.1.4, elementary properties are irreducible (over $\mathbb{L}^a_{\leq}$); this does not hold at all for $+G$, as shown by Lemma 5.1.5, but properties of the form $+G$ are still special, as we will see.

A $\geq$-hereditary property $\mathcal{P}$ is *primitive* if, in every factorisation $\mathcal{P} = \prod_{i\in I}\mathcal{P}_i$ into $\geq$-hereditary properties $\mathcal{P}_i$, there is an $i \in I$ such that $\mathcal{P} = \mathcal{P}_i$. A factorisation $\mathcal{P} = \prod_{i\in I}\mathcal{P}_i$ is *minimal* if there is no $I' \subset I$ such that $\mathcal{P} = \prod_{i\in I'}\mathcal{P}_i$. The set $\min\mathcal{P}$ consists of all $\leq$-minimal elements of $\mathcal{P}$.

5.1.6. Proposition. *Let $\mathcal{P}$ be a $\geq$-hereditary property. Then $\mathcal{P}$ is primitive iff, for some graph G, $\mathcal{P} = +G$. Moreover, $\mathcal{P}$ has a unique minimal factorisation into primitive $\geq$-hereditary properties:*

$$\mathcal{P} = \prod_{G\in\min\mathcal{P}} +G. \qquad (\ddagger)$$

Proof: Since $\mathcal{P} = \bigcup\limits_{G\in\min\mathcal{P}} +G$, $(\ddagger)$ is a valid factorisation of $\mathcal{P}$. Thus, if $\mathcal{P}$ is primitive, then $|\min\mathcal{P}| = 1$. Conversely, let $\min\mathcal{P} = \{G\}$, and consider a factorisation $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots$ of $\mathcal{P}$ into (possibly infinitely many) $\geq$-hereditary properties. If G were not contained in any $\mathcal{P}_i$, then there would have to be graphs $G_1, \ldots, G_r$ in distinct $\mathcal{P}_i$'s, such that $G \in G_1 \circledast \cdots \circledast G_r$; but then each G_i is itself in $\mathcal{P}$, contradicting the minimality of G. So G must be in, say, $\mathcal{P}_1$. By $\geq$-heredity, $\mathcal{P}_1 \supseteq +G = \mathcal{P}$, so $\mathcal{P}_1 = \mathcal{P}$, and thus $\mathcal{P}$ is primitive.

Now let $\mathcal{P}$ be an arbitrary $\geq$-hereditary property, and let $\prod_{G\in S} +G$ be a factorisation into minimal $\geq$-hereditary properties. As argued above, each $G \in \min\mathcal{P}$ must be contained in some property $+H$, where H is in S; by minimality of G, we must have $G = H$. So $\min\mathcal{P} \subseteq S$, and we must have equality for the factorisation to be minimal. $\square$

There is another natural complement of a property $\mathcal{P}$, obtained by taking the complement of all the graphs in $\mathcal{P}$, namely $\overline{\mathcal{P}} := \{\overline{G} \mid G \in \mathcal{P}\}$; for any class $\mathbb{P}$, we define $\overline{\mathbb{P}} := \{\overline{\mathcal{P}} \mid \mathcal{P} \in \mathbb{P}\}$. Clearly $\overline{\overline{\mathcal{P}}} = \mathcal{P}$,

$$\mathcal{F}_{\leq}(\overline{\mathcal{P}}) = \{\overline{H} \mid H \in \mathcal{F}_{\leq}(\mathcal{P})\}$$

and

$$\overline{\mathcal{P}} \circ \overline{\mathcal{Q}} = \overline{\mathcal{P} \circ \mathcal{Q}}.$$

Thus, if $\mathbb{P}$ is closed under multiplication, then so is $\overline{\mathbb{P}}$; moreover, the properties irreducible over $\overline{\mathbb{P}}$ are just the complements of properties irreducible over $\mathbb{P}$, and if $\mathbb{P}$ is uniquely factorisable over itself, then so is $\overline{\mathbb{P}}$.

If $\mathcal{P}$ is induced-hereditary, then so is $\overline{\mathcal{P}}$; so $\overline{\mathbb{L}_{\leq}} = \mathbb{L}_{\leq}$, and it is easy to see that $\overline{\mathbb{L}^{dc}_{\leq}} = \mathbb{L}^{dc}_{\leq}$. However, the only hereditary properties whose graph-wise complements are also hereditary are those of the form $\{G \mid |V(G)| < r\}$, for some finite or infinite cardinal r. To see this, let $\mathcal{P}$ and $\overline{\mathcal{P}}$ be hereditary, and suppose $\mathcal{P}$ contains a graph on n vertices. Then $\overline{K}_n$ is in $\mathcal{P}$, and so K_n is in $\overline{\mathcal{P}}$; thus $\overline{\mathcal{P}}$ contains *all* graphs on n vertices, and so does $\mathcal{P}$.

We note that there is a unique factorisation result for $\mathbb{L}^a \cap \overline{\mathbb{L}^a}$. Let $\mathcal{P}$ be in $\mathbb{L}^a \cap \overline{\mathbb{L}^a}$. Then $\mathcal{P}$ has factorisations into irreducible properties in $\mathbb{L}^a$, and in $\overline{\mathbb{L}^a}$, respectively. Both of these are factorisations in $\mathbb{L}^{dc}_{\leq}$, so they must be the same, and thus the factors are in $\mathbb{L}^a \cap \overline{\mathbb{L}^a}$. Summing up, we have:

5.1.7. Theorem. *Let $\mathbb{P} \in \{\overline{\mathbb{L}^c}, \overline{\mathbb{L}^a}, \mathbb{L}^a \cap \overline{\mathbb{L}^a}, \overline{\mathbb{L}^a_{\leq}}, \mathbb{L}^a_{\leq} \cap \overline{\mathbb{L}^a_{\leq}}\}$. Then properties in $\mathbb{P}$ are uniquely factorisable into irreducible properties in $\mathbb{P}$.* □

5.2 Non-induced-hereditary factors

Szigeti and Tuza [59, Prob. 4] asked whether an additive hereditary property could have a factorisation where the factors are not all additive and hereditary. Semanišin [58] showed that this can happen for properties of the form $\mathcal{P}^2$, where $\mathcal{P}$ contains K_3; we simplify and generalise his result in this section to show, in particular, that we can find a different factorisation for all but the simplest reducible property in $\mathbb{L}^a$ (i.e., $\mathcal{O}^2$, where $\mathcal{O}$ is the set of edgeless graphs). We also show that the remaining cases (irreducible properties and $\mathcal{O}^2$) *are* uniquely factorisable over $\mathbb{U}$.

The alternative factorisations given in Theorem 5.2.1 are obtained by removing K_1 from one of the factors, thus destroying induced-heredity, while retaining additivity. It is easy to check that, if $S \subseteq \mathcal{O}$, then $\mathcal{O}^3 = (\mathcal{O}^2 \setminus S) \circ \mathcal{O}$, so an additive hereditary property can have uncountably many factorisations that involve factors that are neither additive nor induced-hereditary.

If, however, we want to destroy additivity while keeping induced-heredity, we would have to remove some disconnected graph *and all its supergraphs.* It therefore seems plausible that if $\mathcal{P} \in \mathbb{L}^a_{\leq}$ has a factorisation in $\mathbb{L}_{\leq}$, then in fact the factors are in $\mathbb{L}^a_{\leq}$. In other words, that additive induced-hereditary properties are uniquely factorisable *over* $\mathbb{L}_{\leq}$. This much is claimed and stated (with 'induced-hereditary' replaced by 'hereditary') in [51], but we find the argument given there unconvincing, as it centres on the following statements.

> Let $\mathcal{R} = \mathcal{Q}_1 \circ \mathcal{Q}_2 \circ \cdots \circ \mathcal{Q}_m$ be the unique factorisation of the additive hereditary property $\mathcal{R}$ into irreducible hereditary factors. As was shown in the proofs of Theorem 1.1 [quoted here as Theorem 3.2.2] and Theorem 1.2 [see p. 42 of this thesis] they are necessarily additive.

In our opinion, this has not been proved. Our proofs of uniqueness depend heavily on the additivity (or compositivity) of the factors. In fact, when the $\mathcal{Q}_i$'s are all compositive, it does follow that they must be additive (Proposition 5.1.1); but we do not currently see how to show that, if the $\mathcal{Q}_i$'s are hereditary, then they must be compositive.

In light of this problem, and the examples of the preceding section, it is natural to ask how far we can go outside the realm of hereditary compositive and indiscompositive properties before unique factorisation breaks down. This section begins to answer that question.

Recall that, if $\mathbb{P}$ is a class of properties, then a property $\mathcal{P} \in \mathbb{P}$ is *irreducible over* $\mathbb{P}$ if it is not the product of two or more factors that are all in $\mathbb{P}$. The induced-hereditary property $\mathcal{K}$ consists of all the finite complete graphs, while $\mathcal{O}(s) := \{\overline{K_r} \mid 1 \leq r \leq s\}$ and $\mathcal{K}(s) := \{K_r \mid 1 \leq r \leq s\}$. For convenience, we use $\mathcal{O}(\infty)$ and $\mathcal{K}(\infty)$ for $\mathcal{O}$ and $\mathcal{K}$, respectively. We emphasise that in the next result we are considering arbitrary factorisations, not just factorisations into induced-hereditary properties.

5.2.1. Theorem. *Let $\mathcal{P}$ be an induced-hereditary property. Then $\mathcal{P}$ has a unique factorisation if and only if one of the following occurs:*

1. *$\mathcal{P}$ is irreducible;*
2. *$\mathcal{P} = \mathcal{O}(r) \circ \mathcal{O}(s)$, where $r, s \leq \infty$;*
3. *$\mathcal{P} = \mathcal{K}(r) \circ \mathcal{K}(s)$, where $r, s \leq \infty$;*
4. *$\mathcal{P} = \mathcal{O}(r) \circ \mathcal{K}(s)$, where $r, s < \infty$.*

Proof: *Necessity*. We suppose $\mathcal{P}$ is reducible, say $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are not both induced-hereditary, then (by Proposition 2.1.3) $(\mathcal{P}_1)_{\leq} \circ (\mathcal{P}_2)_{\leq}$ is a different factorisation of $\mathcal{P}$. So we need only consider the case where $\mathcal{P}_1$ and $\mathcal{P}_2$ are both in $\mathbb{L}_{\leq}$.

Suppose first that K_2 and $\overline{K_2}$ are both in $\mathcal{P}_1$. Let $\mathcal{P}'_1 = \mathcal{P}_1 \setminus \{K_1\}$. We claim $\mathcal{P} = \mathcal{P}'_1 \circ \mathcal{P}_2$. For if $G \in \mathcal{P}$, let (V_1, V_2) be a $(\mathcal{P}_1, \mathcal{P}_2)$-partition of G. If $|V_1| \neq 1$, then (V_1, V_2) is a $(\mathcal{P}'_1, \mathcal{P}_2)$-partition of G. So suppose $|V_1| = 1$. If $|V_2| = 0$, then $(\emptyset, V_1)$ is a $(\mathcal{P}'_1, \mathcal{P}_2)$-partition of G. Otherwise, let $v \in V_2$. Since both K_2 and $\overline{K_2}$ are in $\mathcal{P}'_1$, and $\mathcal{P}_2$ is induced-hereditary, $(V_1 \cup \{v\}, V_2 \setminus \{v\})$ is a $(\mathcal{P}'_1, \mathcal{P}_2)$-partition of G. In all cases, $G \in \mathcal{P}'_1 \circ \mathcal{P}_2$.

Similarly, if K_2 and $\overline{K_2}$ are both in $\mathcal{P}_2$, then $\mathcal{P}$ does not have a unique factorisation. Now note that, for any induced-hereditary property $\mathcal{Q}$, if $K_2 \notin \mathcal{Q}$, then $\mathcal{Q} \subseteq \mathcal{O}$, while if $\overline{K_2} \notin \mathcal{Q}$, then $\mathcal{Q} \subseteq \mathcal{K}$.

Moreover, since $\mathcal{P}_1$ and $\mathcal{P}_2$ are induced-hereditary, we must be in case (2), (3) or (4), except if $\mathcal{P}_1 = \mathcal{O}(r)$ and $\mathcal{P}_2 = \mathcal{K}$, or $\mathcal{P}_1 = \mathcal{O}$ and $\mathcal{P}_2 = \mathcal{K}(s)$. Note that we may assume that $\mathcal{P}_1 \neq \{K_1\}$ and $\mathcal{P}_2 \neq \{K_1\}$, as otherwise we have either (2) or (3).

Suppose $\mathcal{P}_2 = \mathcal{K}$. We claim $\mathcal{P} = (\mathcal{P}_1 \setminus \{K_1\}) \circ \mathcal{P}_2$. For let $G \in \mathcal{P}$. Then any partition of $V(G)$ into an independent set and a clique works, unless the independent set has size 1. If the independent set has size 1, and its vertex is not joined to every vertex of the clique, then we can make the independent set have size 2. If the independent set has size 1 and is joined to all the vertices in the clique, then G is a clique, and so is in $\mathcal{P}_2$. Similarly, if $\mathcal{P}_1 = \mathcal{O}$, then $\mathcal{P} = \mathcal{P}_1 \circ (\mathcal{P}_2 \setminus \{K_1\})$.

Sufficiency. The sufficiency of (1) is trivial. Let $\mathcal{P} = \mathcal{O}(r) \circ \mathcal{O}(s)$ have some factorisation $\mathcal{Q}_1 \circ \mathcal{Q}_2$. Since $\mathcal{P}$ contains only bipartite graphs, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ contain only edgeless graphs. Let r', s' be positive integers, with $r' \leq r$ if r is finite and $s' \leq s$ if s is finite. $\overline{K_{r'}} + \overline{K_{s'}}$ is in $\mathcal{P}$, and has a unique partition into two independent sets, so we must have $\overline{K_{r'}} \in \mathcal{Q}_1$ and $\overline{K_{s'}} \in \mathcal{Q}_2$ say.

If $r \leq s$ are both finite, then $\overline{K_{r'}} + \overline{K_{r'}}$ shows that both factors contain $\overline{K_{r'}}$ for $r' \leq r$, while $\overline{K_r} + \overline{K_{s'}}$ shows that one factor (and, clearly, only one) contains $\overline{K_{s'}}$ for all $r < s' \leq s$, so $\mathcal{O}(r) \subseteq \mathcal{Q}_1$, $\mathcal{O}(s) \subseteq \mathcal{Q}_2$, and we must clearly have equality. If r or s is infinite, the proof is similar.

The sufficiency of (3) follows from that of (2) by complementation, so we consider $\mathcal{P} = \mathcal{O}(r) \circ \mathcal{K}(s)$; if r or s is 1, then we are in case (2) or (3), so we assume $r, s \geq 2$. Let $\mathcal{Q}_1 \circ \mathcal{Q}_2$ be an arbitrary factorisation of $\mathcal{P}$. If there is a $K_a \in \mathcal{Q}_1$ and a $K_b \in \mathcal{Q}_2$ such that $a \geq 2$ and $b \geq 2$, then $K_a \cup K_b \in \mathcal{P}$, a contradiction. Thus, we can assume $\mathcal{Q}_1$ has no complete graph of size at least 2.

As $K_1 \in \mathcal{O}(r)$ and $K_2 \in \mathcal{K}(s)$, we have $K_3 \in \mathcal{P}$ and, therefore, either K_2 or K_3 is in $\mathcal{Q}_2$. Let $b > 1$ be such that $K_b \in \mathcal{Q}_2$. Then, for every $G \in \mathcal{Q}_1$, $G \cup K_b$ is in $\mathcal{Q}_1 \circ \mathcal{Q}_2$. Let (V_1, V_2) be an $(\mathcal{O}(r), \mathcal{K}(s))$-partition of $G \cup K_b$. Since $b > 1$, at least one vertex from K_b is in V_2. It follows that no vertex from G can be in V_2, so $G \leq (G \cup K_b)[V_1] \in \mathcal{O}(r)$. Hence $\mathcal{Q}_1 \subseteq \mathcal{O}(r)$.

Suppose that $\mathcal{Q}_1 = \{K_1\}$. Since $\overline{K_r} + K_s$ is in $\mathcal{P}$, its $(\{K_1\}, \mathcal{Q}_2)$ partition would imply that $\mathcal{Q}_2$ has a graph containing K_{s+1} or $\overline{K_r} + K_{s-1}$; thus $\mathcal{Q}_1 \circ \mathcal{Q}_2$ has a graph containing K_{s+2}, or a graph containing $\overline{K_r} + (K_{s-1} \cup K_1)$. Since we can check that neither of the last two graphs is in $\mathcal{O}(r) \circ \mathcal{K}(s)$, $\mathcal{Q}_1$ must contain $\overline{K_a}$ for some $a \geq 2$. Now for every $H \in \mathcal{Q}_2$, $\overline{K_a} + H \in \mathcal{O}(r) \circ \mathcal{K}(s)$, and it follows that $\mathcal{Q}_2 \subseteq \mathcal{K}(s)$.

Let $r' \leq r$ be a positive integer. Note that $G = \overline{K_{r'}} + K_s$ is in $\mathcal{O}(r) \circ \mathcal{K}(s)$. There are essentially only two $(\mathcal{O}, \mathcal{K})$-partitions of G and one of these uses K_{s+1}, which is not in $\mathcal{Q}_1$ or $\mathcal{Q}_2$. Therefore, the only possible $(\mathcal{Q}_1, \mathcal{Q}_2)$-partition of $\overline{K_{r'}} + K_s$ shows $\overline{K_{r'}} \in \mathcal{Q}_1$, and thus $\mathcal{Q}_1 = \mathcal{O}(r)$.

Similarly, if $s' \leq s$ is a positive integer, then $\overline{K_r} \cup K_{s'} \in \mathcal{O}(r) \circ \mathcal{K}(s)$. There are essentially only two $(\mathcal{O}, \mathcal{K})$-partitions of G, one of which uses $\overline{K}_{r+1}$, which is not in $\mathcal{Q}_1$. Hence $K_{s'} \in \mathcal{Q}_2$, so $\mathcal{Q}_2 = \mathcal{K}(s)$, as required. $\square$

5.3 Uniquely colourable graphs

We start this section with some results on uniquely partitionable graphs that are either easy or generalise known results. We will then use them to prove facts about generating sets and, in the next section, reducibility and co-primality.

We recall from Section 3.3 that an *unordered* $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$*-partition* of G is a partition $(V_1, \dots, V_m)$ of $V(G)$ such that, for some permutation φ of $\{1, \dots, m\}$ and for each $i \in \{1, \dots, m\}$, V_i is in $\mathcal{Q}_{\varphi(i)}$. If φ is the identity, the partition is *ordered*; an unordered partition may correspond to several ordered ones. Whenever $\mathcal{Q}_i = \mathcal{Q}_j$, interchanging V_i and V_j gives us a different ordered partition; we call this a *trivial interchange*.

A graph G is *uniquely* $(\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_m)$*-partitionable* if it has exactly one unordered $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partition. It is *strongly* uniquely $(\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_m)$-partitionable if, up to trivial interchanges, it has exactly one ordered $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partition. We use 'colour' synonymously with 'partition'.

The following proposition is a bit of a nuisance to prove, but it is important because (using nothing more than induced-heredity) it rules out any freakish behaviour in uniquely partitionable graphs except in a few nice cases that we can specify exactly. It corrects [25, Theorem 1, parts 1,2,4], and we note that Part 5 of the same theorem needs correction too — when $\mathcal{P}_i = \mathcal{O}$ (the edgeless graphs), the term $c(\mathcal{P}_i) + 2$ should be $c(\mathcal{P}_i) + 1$. Any subset of $\mathcal{O}$ or $\mathcal{K}$ (the complete graphs) is irreducible, because it does not contain K_2 or $\overline{K_2}$, respectively.

5.3.1. Proposition. *Let $\mathcal{Q} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$, where the $\mathcal{Q}_i$'s are induced-hereditary properties. Let $G \neq K_1$ be a uniquely $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partitionable graph, with some ordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition $(V_1, V_2, \ldots, V_m)$. Then each V_i is non-empty, and G is $\mathcal{Q}$-strict, unless:*

- *$G = K_r, r < m$, and each $\mathcal{Q}_i$ is contained in $\mathcal{O}$; or*
- *$G = \overline{K_r}, r < m$ and each $\mathcal{Q}_i$ is contained in $\mathcal{K}$; or*
- *$|V(G)| < m$ and each $\mathcal{Q}_i$ is just $\{K_1\}$.*

Proof: Suppose, for some i, that $V_i = \emptyset$. By induced-heredity, we can transfer a vertex from some non-empty V_j to V_i. This gives a different partition, unless $|V_j| = 1$ for all non-empty V_j. In this case, since $G \neq K_1$, we have at least two non-empty sets, say $V_1 = \{v_1\}$ and $V_2 = \{v_2\}$. Suppose v_1 is adjacent to v_2; then K_2 is not in $\mathcal{Q}_1$, $\mathcal{Q}_2$ or $\mathcal{Q}_i$ (otherwise we could transfer v_2 to V_1, or v_1 to V_2, or v_1 and v_2 to V_i), so $\mathcal{Q}_1, \mathcal{Q}_2$, and all $\mathcal{Q}_i$'s with empty parts are contained in $\mathcal{O}$. For any other non-empty part $V_j = \{v_j\}$, we must also have $\mathcal{Q}_j \subseteq \mathcal{O}$, as otherwise we could transfer v_j to V_i, and v_1 and v_2 to V_j.

Similarly, if v_1 is not adjacent to v_2, then all $\mathcal{Q}_i$'s are contained in $\mathcal{K}$. If G is not complete or edgeless, then all $\mathcal{Q}_i$'s must be contained in both $\mathcal{O}$ and $\mathcal{K}$, so each $\mathcal{Q}_i = \{K_1\}$.

Now, assuming each V_i is non-empty, we show that G is $\mathcal{Q}$-strict. Fix a vertex $v_1 \in V_1$; for any $k \neq 1$, moving v_1 to V_k gives a different partition. By induced-heredity, $G[V_1 \setminus \{v_1\}] \in \mathcal{Q}_1$, so it must be that $G[V_k \cup \{v_1\}] \notin \mathcal{Q}_k$. Similarly, fixing a vertex $v_2 \in V_2$, $G[V_1 \cup \{v_2\}] \notin \mathcal{Q}_1$.

So take $G' := G \cup \{v_1'\}$, where v_1' is a copy of v_1, except that the edges between v_1' and V_1 are the same as the edges between v_2 and V_1. Then the unique $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of G cannot be extended to G', so $G' \notin \mathcal{Q}$.

More precisely, if $G' \in \mathcal{Q}$, then there is a permutation φ of $\{1, \dots, m\}$ such that G' has a $(\mathcal{Q}_{\varphi(1)}, \dots, \mathcal{Q}_{\varphi(m)})$-partition $(V_1', \dots, V_m')$, where $G'[V_i'] \in Q_{\varphi(i)}$ and $V_i' \setminus \{v_1'\} = V_i$ for each i. As argued above for the case where φ is the identity, if $v_1' \in V_k'$, $k \neq 1$, then $G'[V_k'] \cong G[V_k \cup \{v_1\}] \notin \mathcal{Q}_{\varphi(k)}$. If $v_1' \in V_1'$, $G'[V_1'] \cong G[V_1 \cup \{v_2\}] \notin \mathcal{Q}_{\varphi(1)}$. □

Broere and Bucko [18] used unique factorisation to determine when uniquely $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partitionable graphs exist. In fact, when such graphs exist we can actually find *strongly* uniquely $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partitionable graphs. These form a generating set not only for the isomorphism classes of graphs in $\mathcal{Q}$, but in fact for all $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-coloured graphs, as we show in Theorem 5.3.2(3) (this was not formally stated by Broere and Bucko).

Let $\mathbb{P}$ be a class of properties. A property $\mathcal{S}$ is a *common divisor* of $\mathcal{P}$ and $\mathcal{Q}$ over $\mathbb{P}$ if, for some (possibly empty) $\mathcal{R}$ and $\mathcal{T}$, $\mathcal{P} = \mathcal{R} \circ \mathcal{S}$ and $\mathcal{Q} = \mathcal{S} \circ \mathcal{T}$, where $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are all in $\mathbb{P}$. $\mathcal{S}$ is a *proper* common divisor if at least one of $\mathcal{R}$ and $\mathcal{T}$ is non-empty. Two properties $\mathcal{P}$ and $\mathcal{Q}$ are *co-prime* over $\mathbb{P}$ if they have no common divisor over $\mathbb{P}$. They are *weakly co-prime* over $\mathbb{P}$ if they have no proper common divisor over $\mathbb{P}$; that is, they are either co-prime over $\mathbb{P}$, or else they have $\mathcal{P} = \mathcal{Q}$ as their only common divisor over $\mathbb{P}$. In the latter case, $\mathcal{P}$ must be irreducible over $\mathbb{P}$; conversely, if $\mathcal{P}$ and $\mathcal{Q}$ are irreducible over $\mathbb{P}$, then they are weakly co-prime over $\mathbb{P}$. If $\mathbb{P} = \mathbb{U}$, we just say that $\mathcal{S}$ is a (proper) common divisor, and $\mathcal{P}$ and $\mathcal{Q}$ are (weakly) co-prime.

5.3.2. Theorem [18]**.** *Let $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ be induced-hereditary properties.*

1. *If there is a uniquely $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partitionable graph $G \neq K_1$, then, for each $i \neq j$, $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are weakly co-prime.*

2. *Let $\mathbb{P}$ be one of $\mathbb{L}^c$, $\mathbb{L}^a$, $\mathbb{L}^a_{\leq}$ and $\mathbb{L}^{dc}_{\leq}$. If each $\mathcal{Q}_i$ is in $\mathbb{P}$ and, for each $i \neq j$, $\mathcal{Q}_i$ and $\mathcal{Q}_j$ are weakly co-prime over $\mathbb{P}$, then there is a strongly uniquely $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partitionable graph $G \neq K_1$.*

3. *With the same hypotheses as in 2, for any $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partition d_0 of some graph G, there is a strongly uniquely $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partitionable graph $H \geq G$ whose unique partition d satisfies $d|G = d_0$.*

Proof: 1. Let $G \neq K_1$ be a graph with a unique $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partition, say $\{V_1, \dots, V_m\}$. If the $\mathcal{Q}_i$'s are all subsets of $\mathcal{O}$ or $\mathcal{K}$, then they are irreducible and thus pairwise weakly co-prime, so the result holds; by Proposition 5.3.1, we can assume that each V_k is non-empty. Suppose for contradiction that, for $i \neq j$, $\mathcal{Q}_i$ and $\mathcal{Q}_j$ have some proper divisor $\mathcal{S}$; say $\mathcal{Q}_i = \mathcal{R} \circ \mathcal{S}$ and $\mathcal{Q}_j = \mathcal{S} \circ \mathcal{T}$, where $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are induced-hereditary, and $\mathcal{T}$ might be empty. By Propositions 5.3.1 and 4.1.3, the $\mathcal{Q}_i$ part must be $\mathcal{Q}_i$-strict, and it thus has non-empty $\mathcal{R}$ and $\mathcal{S}$ sub-parts (by induced-heredity, $K_1 \in \mathcal{R} \cap \mathcal{S}$, so if either sub-part was empty we could put a single vertex in it); similarly, the $\mathcal{Q}_j$ part has non-empty $\mathcal{S}$ sub-part. Then interchanging the $\mathcal{S}$ sub-parts of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ gives a different partition of $V(G)$.

2. We will prove the case $\mathbb{P} = \mathbb{L}_{\leq}^{dc}$, the others being similar and easier. For each i, let $\mathcal{Q}_{i,1} \circ \cdots \circ \mathcal{Q}_{i,t_i}$ be the irreducible factorisation of $\mathcal{Q}_i$ in $\mathbb{L}_{\leq}^{dc}$. If $\mathcal{Q}_{i,j} \subseteq \mathcal{Q}_{i',j'}$, then set $X_{ij;i'j'} := \emptyset$, while if $\mathcal{Q}_{i,j} \nsubseteq \mathcal{Q}_{i',j'}$, then fix $X_{ij;i'j'} \in \mathcal{Q}_{i,j} \setminus \mathcal{Q}_{i',j'}$. Let $H_{i,j}$ be a graph in $\mathcal{Q}_{i,j}$ that contains all $X_{ij;i'j'}$, and let $\mathcal{G}_{i,j}$ be a generating set for $\mathcal{Q}_{i,j}$.

Let $\mathcal{Q} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_m$. Choose a graph $H' \in (\mathcal{G}_{1,1}[H_{1,1}] * \cdots * \mathcal{G}_{m,t_m}[H_{m,t_m}])^{\downarrow}$. By Corollary 4.2.6, there is an induced supergraph H of H' whose unique $\mathcal{Q}$-decomposition with $dec(\mathcal{Q})$ parts is the extension of the obvious (ordered) $(\mathcal{Q}_{1,1}, \ldots, \mathcal{Q}_{m,t_m})$-partition $d = (W_{1,1}, \ldots, W_{m,t_m})$ of H'. Let d_1 be the obvious ordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of H where, for each i, Q_i is obtained by grouping the $\mathcal{Q}_{i,j}$-parts of d. Note that the ind-parts of H form its unique unordered $(\mathcal{Q}_{1,1}, \ldots, \mathcal{Q}_{m,t_m})$-partition.

Now let $d_2 = (V_1, \ldots, V_m)$ be an ordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of H. By Corollary 4.2.3 the ind-parts of H uniformly respect both d_1 and d_2; this means that d_2 is obtained from d_1 by some permutation τ of the ind-parts (that must still give a $(\mathcal{Q}_{1,1}, \ldots, \mathcal{Q}_{m,t_m})$-partition). Now if τ maps a $\mathcal{Q}_{i,j}$ ind-part to a $\mathcal{Q}_{i',j'}$ ind-part we must have, because of $X_{ij;i'j'}$, $\mathcal{Q}_{i,j} \subseteq \mathcal{Q}_{i',j'}$ and, by repeating this argument at most $\sum_{i=1}^{m} t_i$ times, it turns out that $\mathcal{Q}_{i,j} = \mathcal{Q}_{i',j'}$. Then either $i = i'$ or $\mathcal{Q}_i$ and $\mathcal{Q}_{i'}$ are not co-prime in $\mathbb{L}_{\leq}^{dc}$. In the latter case, by hypothesis they must be irreducible over $\mathbb{L}_{\leq}^{dc}$, that is, $\mathcal{Q}_i = \mathcal{Q}_{i,j} = \mathcal{Q}_{i',j'} = \mathcal{Q}_i'$. Thus d_1 and d_2 induce the same ordered $(\mathcal{Q}_1, \ldots, \mathcal{Q}_m)$-partition of $V(H)$, up to trivial interchanges.

3. Let d_0 be $(U_1, U_2, \ldots, U_m)$, and let $(U_{1,1}, \ldots, U_{m,t_m})$ be a $(\mathcal{Q}_{1,1}, \ldots, \mathcal{Q}_{m,t_m})$-partition of G obtained from d_0. By indiscompositivity of the $\mathcal{Q}_{i,j}$'s, we can find a graph H'' that contains G and H' (used in the construction above) as disjoint induced-subgraphs, and that has a partition where the $\mathcal{Q}_{i,j}$ part contains $G[U_{i,j}]$ and $H'[W_{i,j}]$ as disjoint induced-subgraphs. We then take H to be an induced-supergraph of H'' (rather than H'). $\square$

This result has an interesting consequence: a uniquely $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$-partitionable graph exists if and only if there are uniquely $(\mathcal{P}, \mathcal{Q})$-, $(\mathcal{Q}, \mathcal{R})$- and $(\mathcal{R}, \mathcal{P})$-partitionable graphs. However, given three such graphs, it is not clear that there is a direct way of constructing a uniquely $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$-partitionable graph. This is in contrast to the methods of constructing large uniquely $(\mathcal{Q}_1, \dots, \mathcal{Q}_m)$-partitionable graphs from small ones (as in [18, Lemma 3], or the proof of Theorem 5.3.4).

Suppose we have a generating set $\mathcal{G}$ for a property $\mathcal{A} \circ \mathcal{B}$ (where $\mathcal{A}$ and $\mathcal{B}$ are indiscompositive) and for each graph $G \in \mathcal{G}$ we fix an *arbitrary* $(\mathcal{A}, \mathcal{B})$-partition and let $G_A \in \mathcal{A}$ and $G_B \in \mathcal{B}$ be the corresponding subgraphs (so $G \in G_A * G_B$). We can now show that $\{G_A \mid G \in \mathcal{G}\}$ generates $\mathcal{A}$. In other words, we can use a "greedy algorithm" to extract generating sets for $\mathcal{A}$ and $\mathcal{B}$ from a generating set for $\mathcal{A} \circ \mathcal{B}$.

A similar result holds for hereditary compositive properties. Our proof of these two results uses Theorem 5.3.2 and, thus, Unique Factorisation. If there were an elementary proof of the hereditary case, then, by using Lemma 3.1.1, we would have a different proof of Theorem 3.3.1. As it is, we do not even have a direct proof of a special case of Theorem 5.3.3, described below.

For any two hereditary properties $\mathcal{A}$ and $\mathcal{B}$, consider

$$\mathcal{M}_{\mathcal{B}}(\mathcal{A}) := \{G \in \mathcal{A} \mid \exists\, H \in \mathcal{B},\ G + H \in \mathcal{M}(\mathcal{A} \circ \mathcal{B})\}.$$

This is the set of graphs in $\mathcal{A}$ that are extendable to $\mathcal{A}\circ\mathcal{B}$-maximal graphs. By Lemma 3.1.1, $\mathcal{M}_{\mathcal{B}}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, $\mathcal{M}_{\mathcal{A}}(\mathcal{B}) \subseteq \mathcal{M}(\mathcal{B})$, and every $\mathcal{A} \circ \mathcal{B}$-maximal graph is a join of a graph from $\mathcal{M}_{\mathcal{B}}(\mathcal{A})$ and a graph from $\mathcal{M}_{\mathcal{A}}(\mathcal{B})$. Thus $\mathcal{M}_{\mathcal{B}}(\mathcal{A})_{\subseteq} \subseteq \mathcal{A}$, $\mathcal{M}_{\mathcal{A}}(\mathcal{B})_{\subseteq} \subseteq \mathcal{B}$, and $(\mathcal{M}_{\mathcal{B}}(\mathcal{A}) + \mathcal{M}_{\mathcal{A}}(\mathcal{B}))_{\subseteq} = \mathcal{A} \circ \mathcal{B}$; one would expect that the two containments are actually equalities, but we see no way of proving it. However, if $\mathcal{A}$ and $\mathcal{B}$ are hereditary compositive, equality follows from Theorem 5.3.3. Alternatively, we could use Corollary 5.1.4 (which itself depends on Unique Factorisation), and show that $\mathcal{M}_{\mathcal{B}}(\mathcal{A})_{\subseteq}$ and $\mathcal{M}_{\mathcal{A}}(\mathcal{B})_{\subseteq}$ are compositive, but we see no way of doing even that.

5.3.3. Theorem. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{L}^c$. If $(A + B)_{\subseteq} = \mathcal{A} \circ \mathcal{B}$, and $A \subseteq \mathcal{A}$, $B \subseteq \mathcal{B}$, then $A_{\subseteq} = \mathcal{A}$ and $B_{\subseteq} = \mathcal{B}$.* □

5.3.4. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be indiscompositive properties. If $(A * B)_{\leq} = \mathcal{A} \circ \mathcal{B}$, and $A \subseteq \mathcal{A}$, $B \subseteq \mathcal{B}$, then $A_{\leq} = \mathcal{A}$ and $B_{\leq} = \mathcal{B}$.*

Proof: Let $\mathcal{A} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_r$ and $\mathcal{B} = \mathcal{P}_{r+1} \circ \cdots \circ \mathcal{P}_s$ be the factorisations of $\mathcal{A}$ and $\mathcal{B}$ into irreducible indiscompositive factors.

Let G be an arbitrary graph in $\mathcal{A}$, with some $(\mathcal{P}_1, \ldots, \mathcal{P}_r)$-partition $(U_1, \ldots, U_r)$. Let Φ be the set of 1-1 mappings $\varphi : \{1, \ldots, r\} \to \{1, \ldots, s\}$ such that, for each $i \in \{1, \ldots, r\}$, $\mathcal{P}_i = \mathcal{P}_{\varphi(i)}$. We construct a graph $G' \in \mathcal{A} \circ \mathcal{B}$ with $|\Phi|$ disjoint copies of G: for each $\varphi \in \Phi$, there is a copy $G_\varphi \leq G'$ which has, for each $i \in \{1, \ldots, r\}$, a copy of U_i in $\mathcal{P}_{\varphi(i)}$. Such a graph exists by indiscompositivity of the $\mathcal{P}_i$'s.

Let d_0 be the obvious $(\mathcal{P}_1, \ldots, \mathcal{P}_s)$-partition of G'. By Theorem 5.3.2(3), there is a graph $H \geq G'$ with a $(\mathcal{P}_1, \ldots, \mathcal{P}_s)$-partition $d = (V_1, \ldots, V_s)$ that is unique up to trivial interchanges, and such that $d|G' = d_0$.

Now find a graph $H' \in A * B$ such that $H \leq H'$. Suppose $H' \in H'_A * H'_B$, where $H'_A \in A \subseteq \mathcal{A}$ and $H'_B \in B \subseteq \mathcal{B}$. Then there is a $(\mathcal{P}_1, \ldots, \mathcal{P}_s)$-partition $(W_1, \ldots, W_s)$ of H', such that $H'_A = H'[W_1 \cup \cdots \cup W_r]$.

Because H is strongly uniquely partitionable, there is a $1-1$ mapping φ such that, for for each $i \in \{1, \ldots, r\}$, $V_i \leq W_{\varphi(i)}$ and $\mathcal{P}_i = \mathcal{P}_{\varphi(i)}$. Then $G_\varphi \leq H'_A$, so $G \in A_{\leq}$. Since G was arbitrary, $A_{\leq} = \mathcal{A}$. Similarly, $B_{\leq} = \mathcal{B}$. $\Box$

5.4 Irreducibility and co-primality

In this section we show that we can talk unambiguously about properties in $\mathbb{P}$ that are irreducible or co-prime, as these are exactly the properties that are irreducible over $\mathbb{P}$ or co-prime over $\mathbb{P}$.

5.4.1. Theorem. *Let $\mathbb{P}$ be in $\{\mathbb{L}, \mathbb{L}_{\leq}, \mathbb{L}^c, \mathbb{L}^a, \mathbb{L}^a_{\leq}, \mathbb{L}^{dc}_{\leq}\}$ and let $\mathcal{P}, \mathcal{Q}$ be in $\mathbb{P}$. Then*

A. $\mathcal{P}$ is irreducible iff it is irreducible over $\mathbb{P}$;

B. $\mathcal{P}$ and $\mathcal{Q}$ are co-prime iff they are co-prime over $\mathbb{P}$.

Proof: Consider first $\mathbb{P} = \mathbb{L}$ (the proofs for $\mathbb{L}_{\leq}$ are similar). Suppose $\mathcal{P} \in \mathbb{L}$ is reducible, say $\mathcal{P} = \mathcal{Q} \circ \mathcal{R}$. Then, by Proposition 2.1.3, $\mathcal{P}_{\subseteq} = \mathcal{P} = \mathcal{Q} \circ \mathcal{R} \subseteq \mathcal{Q}_{\subseteq} \circ \mathcal{R}_{\subseteq} = (\mathcal{Q} \circ \mathcal{R})_{\subseteq} = \mathcal{P}_{\subseteq}$, so we have equality throughout; in particular $\mathcal{P}$ is the product of $\mathcal{Q}_{\subseteq}$ and $\mathcal{R}_{\subseteq}$, which are both in $\mathbb{L}$.

For part B, if $\mathcal{P} = \mathcal{R} \circ \mathcal{S}$ and $\mathcal{Q} = \mathcal{S} \circ \mathcal{T}$, then $\mathcal{P} = \mathcal{R}_{\subseteq} \circ \mathcal{S}_{\subseteq}$ and $\mathcal{Q} = \mathcal{S}_{\subseteq} \circ \mathcal{T}_{\subseteq}$, so $\mathcal{P}$ and $\mathcal{Q}$ have a common divisor in $\mathbb{L}$.

The additive and indiscompositive analogues of Proposition 2.1.3 are not true ($\mathcal{P} = \mathcal{Q} = \{K_1\}$ is a counterexample for both parts), and thus the proof of Theorem 5.4.1 for $\mathbb{P} \in \{\mathbb{L}^c, \mathbb{L}^a, \mathbb{L}^a_\leq, \mathbb{L}^{dc}_\leq\}$ is not so elementary. In fact it depends on Theorem 5.3.2, but see Theorem 3.2.3 for part A, when $\mathbb{P} \in \{\mathbb{L}^c, \mathbb{L}^a\}$.

If $\mathcal{P}$ is irreducible over $\mathbb{P}$, then by Theorem 5.3.2(2) there is a uniquely $(\mathcal{P}, \mathcal{P})$-partitionable graph. By Theorem 5.3.2(1), either $\mathcal{P}$ and $\mathcal{P}$ are co-prime or both are irreducible. This implies $\mathcal{P}$ is irreducible.

If $\mathcal{P}$ and $\mathcal{Q}$ are co-prime over $\mathbb{P}$, then by Theorem 5.3.2(2), there is a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graph. Therefore, Theorem 5.3.2(1) implies that either $\mathcal{P}$ and $\mathcal{Q}$ are co-prime or both are irreducible. If both are irreducible, then the fact that $\mathcal{P}$ and $\mathcal{Q}$ are co-prime over $\mathbb{P}$ tells us that $\mathcal{P} \neq \mathcal{Q}$, so they are in any case co-prime. $\square$

We can now re-state Theorems 3.3.3 and 4.3.7 as follows.

5.4.2. Unique Factorisation Theorems. *Let $\mathbb{P}$ be in $\{\mathbb{L}^c, \mathbb{L}^a, \mathbb{L}^a_\leq, \mathbb{L}^{dc}_\leq\}$. A property in $\mathbb{P}$ is uniquely factorisable into irreducible properties that are in $\mathbb{P}$, and the number of factors is exactly $dc(\mathcal{P})$ (for $\mathcal{P} \in \mathbb{L}^c, \mathbb{L}^a$) or $dec(\mathcal{P})$ (for $\mathcal{P} \in \mathbb{L}^a, \mathbb{L}^a_\leq, \mathbb{L}^{dc}_\leq$).* $\square$

Chapter 6

Complexity

Can the vertices of a graph G be partitioned into $A \cup B$, so that $G[A]$ is a line-graph and $G[B]$ is a forest? Can G be partitioned into a planar graph and a perfect graph? The NP-completeness of these problems are just special cases of the main result proved in this chapter:

Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be additive induced-hereditary properties, $\mathcal{P} \circ \mathcal{Q} \neq \mathcal{O}^2$. Then $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is NP-hard. Moreover, it is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both in NP.* □

A lot of the background and history related to this result was given in Section 1.4. The computational problem we are considering, also known as $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-PARTITIONING or $(\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n)$-RECOGNITION, is the following:

$(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-COLOURING
Instance: a finite simple graph G.
Problem: is there a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colouring of G?

We recall three main advances in the study of this problem's complexity:

- the Hell-Nešetřil result [42] that hom-properties are NP-complete to recognise, with the exception of $\mathcal{O}$ and $\mathcal{O}^2$;

- the result of Achlioptas [1] that $\mathcal{P}^k$-RECOGNITION is NP-hard when $k \geq 2$ and $\mathcal{P} \neq \mathcal{O}$ is an irreducible additive hereditary property; and

- the proof by Kratochvíl and Schiermeyer [48] that $(\mathcal{O}, \mathcal{P})$-COLOURING is NP-hard for any additive *hereditary* $\mathcal{P} \neq \mathcal{O}$.

Kratochvíl and Schiermeyer conjectured that $(\mathcal{P}, \mathcal{Q})$-COLOURING is NP-hard for *any* additive hereditary $\mathcal{P}$ and $\mathcal{Q}$, except if $\mathcal{P} = \mathcal{Q} = \mathcal{O}$. Our result not only settles this conjecture, but also shows it to hold when $\mathcal{P}$ and $\mathcal{Q}$ are additive *induced*-hereditary.

On the other hand, the Hell-Nešetřil result, along with Kratochvíl and Mihók's characterisation of reducible hom-properties [47], shows that even irreducible additive hereditary properties can be NP-complete to recognise.

Note that $\mathcal{O}$ is the smallest additive induced-hereditary property. Since $\mathcal{O}^k$-COLOURING is well-known to be NP-complete when $k \geq 3$, one could reasonably expect to find a quick proof that $\mathcal{P}^k$-COLOURING is NP-hard, for any additive induced-hereditary property $\mathcal{P}$. There is, in fact, an easy transformation, due to Achlioptas, when $\mathcal{P}$ is irreducible. Although we do not require the proof, it is somewhat instructive.

6.1.1. Theorem [1]**.** *Let $\mathcal{P}$ be an irreducible additive induced-hereditary property. Then, for every $k \geq 3$, $\mathcal{P}^k$-COLOURING is NP-hard.*

Proof: We will transform from GRAPH k-COLOURING.

First we construct a gadget H with two special vertices v_1 and v_2, such that, in any $(\mathcal{P}, \ldots, \mathcal{P})$-partition, v_1 and v_2 must be in different parts, say V_1 and V_2, and, for $i = 1, 2$, v_i is not adjacent to any vertex of $V_i \setminus \{v_i\}$. Because $\mathcal{P}$ is irreducible, there is a uniquely $(\mathcal{P}, \ldots, \mathcal{P})$-partitionable graph H', by Theorem 5.3.2. Let its unique partition be $(U_1, \ldots, U_k)$, and fix $u_1 \in U_1$. Add a vertex v_1 that is adjacent to the vertices in $N(u_1) \setminus U_1$. By additivity of $\mathcal{P}$, $(U_1 \cup \{v_1\}, U_2, \ldots, U_k)$ is a $(\mathcal{P}, \ldots, \mathcal{P})$-partition of the new graph H''; in fact, it is its only partition, because if v_1 could be put with U_j, $j \neq 1$, then in H' we could have put u_1 with U_j, giving a different partition. Similarly, we add a vertex v_2 that must go with U_2, but is not adjacent to any vertex of U_2. The resulting graph is H, where, for $i = 1, 2$, we let $V_i := U_i \cup \{v_i\}$.

Now, for an arbitrary graph G, we create G' by replacing each edge xy with a copy of H, identifying x and y with v_1 and v_2 respectively. If G' has a $(\mathcal{P}, \ldots, \mathcal{P})$-partition, then the ends of every edge from G must receive different colours, giving us a k-colouring of G. Conversely, a k-colouring of G gives different colours to the copies of v_1 and v_2 in any copy of H in G'. We can extend this to separate $(\mathcal{P}, \ldots, \mathcal{P})$-colourings of each copy of H; this is actually a $(\mathcal{P}, \ldots, \mathcal{P})$-colouring of all of G', since each colour class in G' is a union of components from the colour-classes of copies of H. $\square$

This proof, like virtually all other complexity proofs in generalised colourings, shows the crucial importance of uniquely partitionable graphs. It also illustrates three key difficulties.

- The proof works only for irreducible properties, as for reducible properties there may not be any uniquely colourable graphs.
- Transformations from GRAPH COLOURING are only useful for generalised colourings that involve three or more (possibly equal) properties. For two properties, a different NP-complete problem is needed.
- The proof technique used above works well when the properties are all identical; if we were to use it for $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring, we would artificially force some vertices of the graph G' to be in particular $\mathcal{P}_i$'s.

To deal with the first problem, we will prove that, given additive induced-hereditary properties $\mathcal{P}$ and $\mathcal{Q}$, $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is at least as hard as $\mathcal{P}$-RECOGNITION. While it seems intuitively obvious, this result is not true if $\mathcal{Q}$ is additive but not induced-hereditary, e.g., consider $\mathcal{Q} := \{G \mid |V(G)| \geq 10\}$. Kratochvíl and Schiermeyer proved a special case of this in [48].

We will actually prove a slightly stronger result. An induced-hereditary property $\mathcal{P}$ is *polynomially indiscompositive* if, given a fixed graph $H \in \mathcal{P}$, there is an algorithm that, for any graph $G \in \mathcal{P}$ finds, in time polynomial in $|V(G)|$, an appropriate set of edges to put between G and H to produce a graph $G_H \in \mathcal{P}$ containing G and H as disjoint induced-subgraphs. The class of polynomially indiscompositive properties is the largest class for which our current proof techniques work; happily for us, it includes both $\mathbb{L}^a_{\leq}$ and $\overline{\mathbb{L}^a_{\leq}}$, allowing us to extend Theorem 6.1.4 to Theorems 6.1.5 and 6.1.6.

Note that it is not enough to have the graph G_H — we need to know (in polynomial time) which vertices correspond to G and which correspond to H. Also note that, if we run the algorithm on a graph $G \notin \mathcal{P}$, then it will either exceed its time-bound, or give a set of edges that produces a graph G_H that is not in $\mathcal{P}$ (because it contains G); so G_H is in $\mathcal{P}$ iff G is in $\mathcal{P}$.

6.1.2. Theorem. *Let $\mathcal{P}$ be polynomially indiscompositive, and let $\mathcal{Q}$ be indiscompositive. Then there is a polynomial-time transformation from the $\mathcal{P}$-RECOGNITION problem to the $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION problem.*

Proof: For any graph G we will construct a graph $\widetilde{G}$ such that $G \in \mathcal{P}$ if and only if $\widetilde{G} \in \mathcal{P} \circ \mathcal{Q}$ (see Figure 6.1).

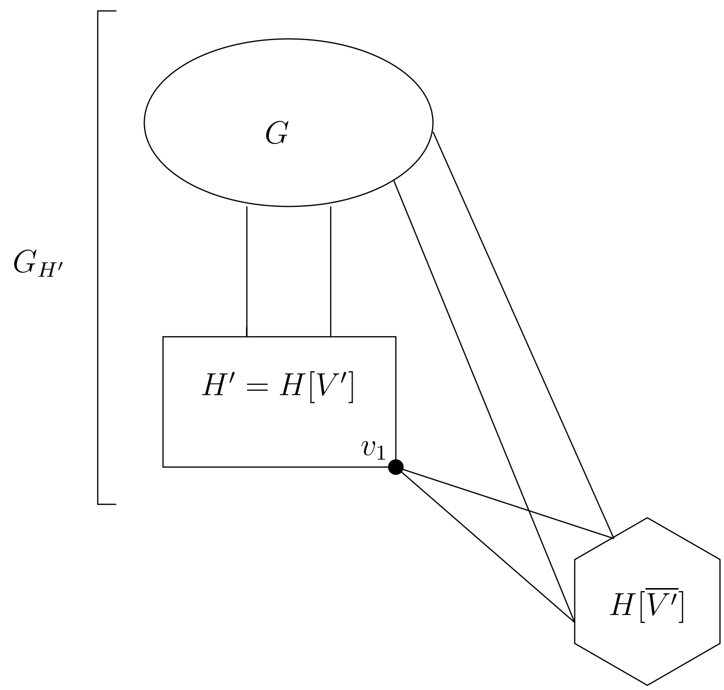


Figure 6.1: Constructing $\widetilde{G}$ — the induced-hereditary case.

Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ and $\mathcal{Q} = \mathcal{P}_{n+1} \circ \cdots \circ \mathcal{P}_{n+r}$, where the $\mathcal{P}_i$'s are irreducible indiscompositive properties. By Theorem 5.3.2 there is a strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_{n+r})$-partitionable graph H with partition $(V_1, \ldots, V_{n+r})$. By Lemma 5.3.1 we can assume that each V_i is non-empty. Let v_1 be some fixed vertex in V_1, and define $V' := V_1 \cup \cdots \cup V_n$, $\overline{V'} := V_{n+1} \cup \cdots \cup V_{n+r}$ and $H' := H[V']$.

We construct $\widetilde{G}$ by making every vertex of G adjacent to every vertex of $N(v_1) \cap \overline{V'}$, and adding edges so that $V(G) \cup V'$ induces a graph $G_{H'}$ that is in $\mathcal{P}$ iff G is in $\mathcal{P}$. We can do this in time polynomial in $|V(G)|$.

Clearly, if G is in $\mathcal{P}$, then $\widetilde{G}$ is in $\mathcal{P} \circ \mathcal{Q}$. Conversely, if $\widetilde{G} \in \mathcal{P} \circ \mathcal{Q}$, then it has an ordered partition $(W_1, \ldots, W_{n+r})$ with $W_i \in \mathcal{P}_i$ for each i. Since the $\mathcal{P}_i$'s are induced-hereditary, $\widetilde{G}[W_i] \in \mathcal{P}_i$ implies $\widetilde{G}[W_i \cap V(H)] \in \mathcal{P}_i$. Then, up to trivial interchanges, $(W_1 \cap V(H), \ldots, W_{n+r} \cap V(H)) = (V_1, \ldots, V_{n+r})$; in particular, $v_1 \in W_1$.

Suppose there is some $w \in V(G)$ and some $i \geq 1$ such that $w \in W_{n+i}$; without loss of generality, $i = r$. Then $\widetilde{G}[V_{n+r} \cup \{w\}] \cong H[V_{n+r} \cup \{v_1\}]$ is in $\mathcal{P}_{n+r}$, so $(V_1 \setminus \{v_1\}, V_2, \ldots, V_{n+r-1}, V_{n+r} \cup \{v_1\})$ is a new $(\mathcal{P}_1, \ldots, \mathcal{P}_{n+r})$-partition of H (since $V_1 \neq \emptyset \neq V_{n+r}$), a contradiction.

Thus no vertex of G is in W_{n+i}, for any $i \geq 1$, and so $G \leq \widetilde{G}[W_1 \cup \cdots \cup W_n] \in \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n = \mathcal{P}$, and $G \in \mathcal{P}$ as required. $\square$

The hereditary compositive version of Theorem 6.1.2 is stronger, because now $G_{H'}$ can *always* be found in polynomial time. G and H' might share some common vertices in the graph $G_{H'}$, in which case two vertices are joined by an edge if they were adjacent either in G or in H'. But, since $\mathcal{P}$ is hereditary, there is no point in adding any extra edges to get a graph in $\mathcal{P}$. So we just need to try out all $\binom{|V(G)|}{s}$ possible sets of vertices from G to overlap with H', where $0 \leq s \leq |V(H')|$. Having overlapped G with H' to obtain a potential $G_{H'}$, we take the join of $G_{H'}$ and $H[\overline{V'}]$ (see Figure 6.2) and run the $\mathcal{P} \circ \mathcal{Q}$-recognition algorithm on the resulting $\widetilde{G}$. The graph G is in $\mathcal{P}$ iff at least one of the (polynomially many) possible $\widetilde{G}$ is in $\mathcal{P} \circ \mathcal{Q}$.

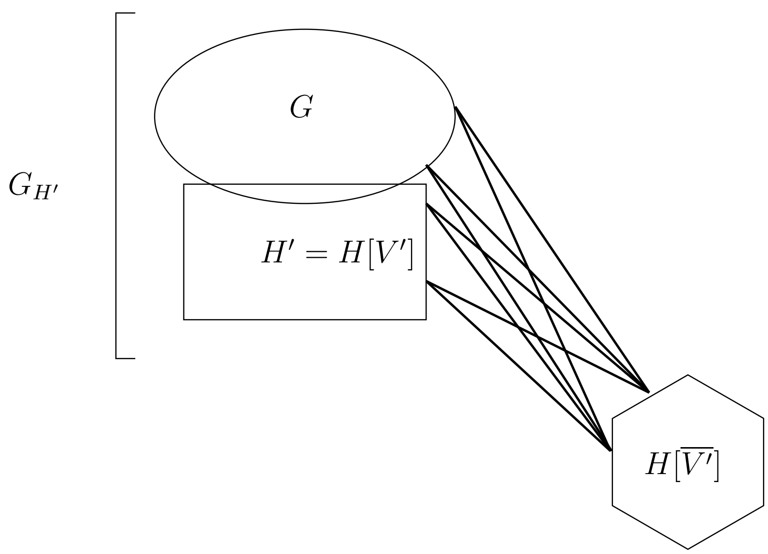


Figure 6.2: Constructing $\widetilde{G}$ — the hereditary case.

6.1.3. Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary compositive properties. Then the $\mathcal{P}$-*RECOGNITION *problem is polynomial-time reducible to the $\mathcal{P} \circ \mathcal{Q}$-*RECOGNITION *problem.* $\square$

Theorem 6.1.2 immediately shows that Theorem 6.1.1 holds even without the irreducibility assumption. In fact, any additive induced-hereditary property of the form $\mathcal{O}^3 \circ \mathcal{Q}$ or $\mathcal{P}^2 \circ \mathcal{Q}$ (where $\mathcal{P} \neq \mathcal{O}$) is now seen to be NP-hard to recognise. To establish this for all reducible additive induced-hereditary properties (except $\mathcal{O}^2$) we need a different proof, and a computational problem different from graph colouring.

The NP-complete problem of choice is p-IN-r-SAT. Kratochvíl and Schiermeyer, and Achlioptas, used a variant called NOT-ALL-EQUAL-SAT, which is just (BETWEEN-1-AND-$(r-1)$)-IN-r-SAT. Schaefer [54] showed p-IN-r-SAT and its variants to be NP-complete, even for formulae with all literals unnegated, for any fixed p and r, so long as $0 < p < r$ and $r \geq 3$. We restate it as:

p-IN-r-COLOURING
Instance: an r-uniform hypergraph.
Problem: is there a set U of vertices such that, for each hyper-edge e, $|U \cap e| = p$?

6.1.4. Theorem. *Let $\mathcal{P}$ and $\mathcal{Q}$ be additive induced-hereditary properties, $\mathcal{P} \circ \mathcal{Q} \neq \mathcal{O}^2$. Then $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is NP-hard. Moreover, it is NP-complete iff $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both in NP.*

Proof: We will prove the first part. The second part then follows easily — if $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both in NP, then clearly $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is in NP, while if $\mathcal{P}$- or $\mathcal{Q}$-RECOGNITION is not in NP, then, by Theorem 6.1.2, $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is not in NP.

Also by Theorem 6.1.2 (and by the well-known NP-hardness of recognising $\mathcal{O}^3$ [44]), we need only consider the case where $\mathcal{P}$ and $\mathcal{Q}$ are irreducible to prove the first part. By Theorem 5.3.2 there is a strongly uniquely $(\mathcal{P}, \mathcal{Q})$-colourable graph $G_{\mathcal{P},\mathcal{Q}}$ that we use to "force" vertices to be in $\mathcal{P}$ or $\mathcal{Q}$.

More formally, let the unique partition be $V(G_{\mathcal{P},\mathcal{Q}}) = U_{\mathcal{P}} \cup U_{\mathcal{Q}}$. Choose $p \in U_{\mathcal{P}}$. For any graph H, if $G_{\mathcal{P},\mathcal{Q}} \leq H$, and $v \notin V(G_{\mathcal{P},\mathcal{Q}})$ satisfies $N(v) \cap U_{\mathcal{Q}} = N(p) \cap U_{\mathcal{Q}}$, then in any $(\mathcal{P}, \mathcal{Q})$-colouring of H, v must be in the $\mathcal{P}$-part[1]; otherwise, in $G_{\mathcal{P},\mathcal{Q}}$ we could transfer p over to the $\mathcal{Q}$ part,

[1]To be precise, we mean that v is coloured the same as p: if $\mathcal{P} = \mathcal{Q}$ then a $(\mathcal{P}, \mathcal{Q})$-colouring is also a $(\mathcal{Q}, \mathcal{P})$-colouring, but we adopt the convention that the $\mathcal{P}$-part is the part containing p.

giving us a different $(\mathcal{P}, \mathcal{Q})$-colouring. Similarly we choose $q \in U_{\mathcal{Q}}$, whose neighbours we use to force vertices to be in $\mathcal{Q}$. $G_{\mathcal{P},\mathcal{Q}}$ is our first gadget.

An *end-block* of a graph G is a block of G that contains at most one cut-vertex of G; if G has no cut-vertices, then G is itself an end-block. Let $B_{\mathcal{P}}$ be an end-block of $F_{\mathcal{P}} \in \mathcal{F}_{\leq}(\mathcal{P})$, chosen to have the least number of vertices among all the end-blocks of all the graphs in $\mathcal{F}_{\leq}(\mathcal{P})$ (see Figure 6.3). Because $\mathcal{P}$ is additive and non-empty, $F_{\mathcal{P}}$ is connected and has at least two vertices, so $B_{\mathcal{P}}$ has $k \geq 2$ vertices. The point to note is that, if H is a graph in $\mathcal{P}$, then adding an end-block with fewer than k vertices produces another graph in $\mathcal{P}$.

Let $y_{\mathcal{P}}$ be the unique cut-vertex contained in $B_{\mathcal{P}}$ (if $B_{\mathcal{P}} = F_{\mathcal{P}}$, pick $y_{\mathcal{P}}$ arbitrarily), and let $x_{\mathcal{P}}$ be a vertex of $B_{\mathcal{P}}$ adjacent to $y_{\mathcal{P}}$. Let $F'_{\mathcal{P}}$ be the graph obtained by adding an extra copy of $B_{\mathcal{P}}$ (incident to the same cut-vertex $y_{\mathcal{P}}$), and let $x'_{\mathcal{P}}$ be a vertex in this new copy that is adjacent to $y_{\mathcal{P}}$.

Similarly, we choose $B_{\mathcal{Q}}$ to be an end-block of $F_{\mathcal{Q}} \in \mathcal{F}_{\leq}(\mathcal{Q})$, minimal among the end-blocks of graphs in $\mathcal{F}_{\leq}(\mathcal{Q})$; we add a copy of $B_{\mathcal{Q}}$, and pick $x_{\mathcal{Q}}$, $y_{\mathcal{Q}}$ and $x'_{\mathcal{Q}}$ as above. We identify $x_{\mathcal{P}}$ with $x_{\mathcal{Q}}$, $y_{\mathcal{P}}$ with $y_{\mathcal{Q}}$, $x'_{\mathcal{P}}$ with $x'_{\mathcal{Q}}$, and label the identified vertices x, y, x'.

Finally, we force all the vertices of $F'_{\mathcal{P}}$ (except for x, y, x') to be in $\mathcal{P}$, and all the vertices of $F'_{\mathcal{Q}}$ (except for x, y, x') to be in $\mathcal{Q}$. That is, we add a copy of $G_{\mathcal{P},\mathcal{Q}}$, and make every vertex of $F'_{\mathcal{P}} - \{x, y, x'\}$ adjacent to every vertex of $N(p) \cap U_{\mathcal{Q}}$, and every vertex of $F'_{\mathcal{Q}} - \{x, y, x'\}$ adjacent to every vertex of $N(q) \cap U_{\mathcal{P}}$ (in Figure 6.3, the vertices of $F'_{\mathcal{P}}$ and $F'_{\mathcal{Q}}$ are shaded the same as their neighbours in $G_{\mathcal{P},\mathcal{Q}}$, but they have the opposite colour).

It can be readily checked that the resulting gadget R (for ‘replicator’) has the following properties:

- In a $(\mathcal{P}, \mathcal{Q})$-colouring of R, x and x' must have the same colour; moreover, there is at least one colouring (in fact, exactly one) in which x and x' are in $\mathcal{P}$, and at least one in which both are in $\mathcal{Q}$.

  To see this, note that the colour of every vertex not in $\{x, y, x'\}$ is fixed; and y cannot have the same colour as x or x', because if, say, y and x were both in $\mathcal{P}$, then we would have a copy of $F_{\mathcal{P}}$ in $\mathcal{P}$. So if y is in $\mathcal{P}$, then x and x' must be in $\mathcal{Q}$ (and *vice versa*). This is a valid colouring because each component in the $\mathcal{Q}$ part is a proper subgraph of $F_{\mathcal{Q}}$, while the $\mathcal{P}$ part is obtained from a proper subgraph of $F_{\mathcal{P}}$ by attaching two end-blocks (each with one fewer vertex than $B_{\mathcal{P}}$).

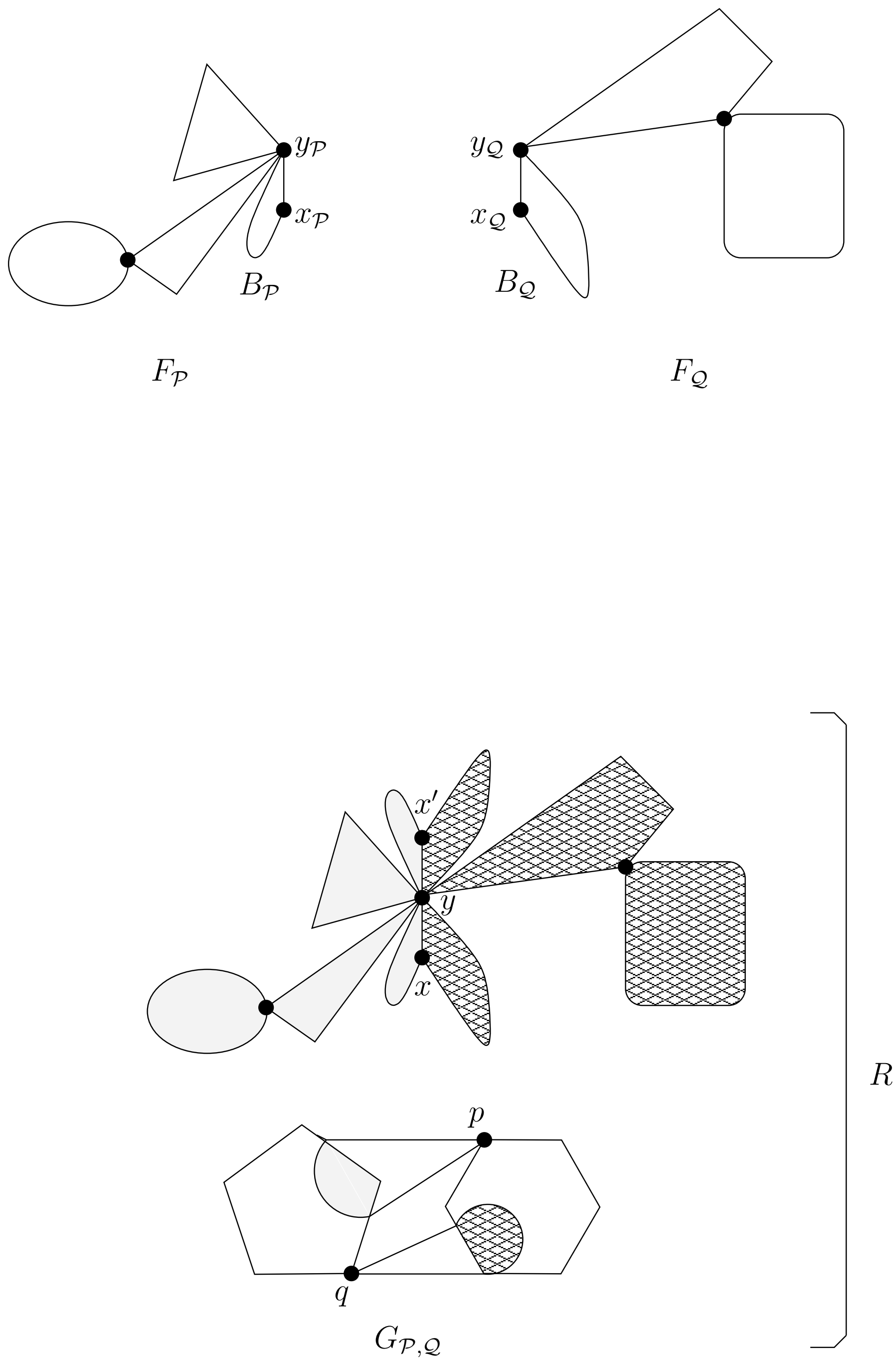


Figure 6.3: The forbidden graphs $F_{\mathcal{P}}$ and $F_{\mathcal{Q}}$, and the replicator gadget R.

- Identify x with a vertex z of some graph H to obtain H_R. Then $(\mathcal{P}, \mathcal{Q})$-colourings of H and R that agree on x together give a $(\mathcal{P}, \mathcal{Q})$-colouring of H_R. If x is in $\mathcal{Q}$, say, then the $\mathcal{Q}$-part of H_R is the $\mathcal{Q}$-part of H, with a small end-block $(B_{\mathcal{Q}} - y)$ attached, and some components from the $\mathcal{Q}$-part of R. This is actually a graph in $\mathcal{Q}$, while the $\mathcal{P}$-part is the disjoint union of the $\mathcal{P}$-parts of H and R.

  Similarly, we can identify x' with some vertex z' of a graph H', and attach more copies of R at x or x'.

We thus have a gadget that "replicates" the colour of x on x', while preserving valid colourings.

Let $H_{\mathcal{P}}$ be a forbidden subgraph for $\mathcal{P}$ with the least possible number of vertices, say $p+1$; similarly choose $H_{\mathcal{Q}} \in \mathcal{F}_{\leq}(\mathcal{Q})$ on $q+1$ vertices, where $q+1$ is as small as possible, so any graph on at most p (resp. q) vertices is in $\mathcal{P}$ (resp. $\mathcal{Q}$). Note that $H_{\mathcal{P}}$ is connected because $\mathcal{P}$ is additive, and that $\mathcal{P} = \mathcal{O}$ iff $H_{\mathcal{P}} = K_2$ iff $p+1 = 2$. Since $\mathcal{P}$ and $\mathcal{Q}$ are not both $\mathcal{O}$, $p+q \geq 3$, and so p-IN-$(p+q)$-COLOURING is NP-complete. We will construct a third gadget to transform this to $(\mathcal{P}, \mathcal{Q})$-COLOURING.

We start with an independent set S on $p+q$ vertices, $\{x_1, \ldots, x_{p+q}\}$. For every $(p+1)$-subset of S, say $T_j = \{x_1, \ldots, x_{p+1}\}$, add a disjoint copy of $H_{\mathcal{P}}$ whose vertices are labeled $x_1^j, \ldots, x_{p+1}^j$. For each $i = 1, \ldots, p+1$, use a new copy $R_{i,j}$ of R to ensure that x_i and x_i^j are always coloured the same; to do this, identify the vertices x and x' of $R_{i,j}$ with x_i and x_i^j. For every $(q+1)$-subset of S we add a copy of $H_{\mathcal{Q}}$ in the same manner. Thus every vertex $x_i \in S$ will have $\ell = \binom{p+q-1}{p} + \binom{p+q-1}{q}$ 'shadow vertices' $x_i^1, \ldots, x_i^\ell$ from copies of $H_{\mathcal{P}}$ and $H_{\mathcal{Q}}$. Call this gadget N (for 'pin cushion' — the copies of $H_{\mathcal{P}}$ and $H_{\mathcal{Q}}$ being stuck into the independent set S by 'pins' or 'replicators').

In a $(\mathcal{P}, \mathcal{Q})$-colouring of N, no $p+1$ vertices of S can be in $\mathcal{P}$, and no $q+1$ vertices can be in $\mathcal{Q}$, so exactly p vertices of S are in $\mathcal{P}$, and exactly q are in $\mathcal{Q}$. Conversely, suppose that exactly p vertices of S are coloured red, and the other q are blue; colour each vertex x_i^j the same as x_i, $1 \leq i \leq p+q$, $1 \leq j \leq \ell$. Then each copy of $H_{\mathcal{P}}$ or $H_{\mathcal{Q}}$ has at most p red and at most q blue vertices, giving it a valid $(\mathcal{P}, \mathcal{Q})$-colouring. The colouring on the rest of each gadget $R_{i,j}$ is then forced, and we have a $(\mathcal{P}, \mathcal{Q})$-colouring of all of N.

Now, given a $(p+q)$-uniform hypergraph $\mathcal{H}$, we take $|E(\mathcal{H})|$ copies of N, identifying each copy's independent set S with a distinct hyperedge. The resulting graph $\mathcal{H}'$ is $(\mathcal{P},\mathcal{Q})$-colourable iff $\mathcal{H}$ has a p-in-$(p+q)$-colouring. Note that $\mathcal{P},\mathcal{Q},p$ and q are all fixed, as are our three gadgets; it is only the hypergraph $\mathcal{H}$ that is arbitrary, and since $\mathcal{H}'$ can be constructed in time that is linear in $|E(\mathcal{H})|$, we are done. $\square$

Recall from Section 5.1 that the graphwise complement of a property $\mathcal{P}$ is $\overline{\mathcal{P}} := \{\overline{G} \mid G \in \mathcal{P}\}$, while for a class $\mathbb{P}$, we define $\overline{\mathbb{P}} := \{\overline{\mathcal{P}} \mid \mathcal{P} \in \mathbb{P}\}$. Given two classes $\mathbb{P}_1$ and $\mathbb{P}_2$, the class $\mathbb{P}_1 \circ \mathbb{P}_2$ is $\mathbb{P}_1 \cup \mathbb{P}_2 \cup \{\mathcal{P}_1 \circ \mathcal{P}_2 \mid \mathcal{P}_1 \in \mathbb{P}_1, \mathcal{P}_2 \in \mathbb{P}_2\}$.

A property is *co-additive* if it is in $\overline{\mathbb{L}^a_\leq}$. Note that $\mathcal{K} = \overline{\mathcal{O}} \in \overline{\mathbb{L}^a_\leq}$. The complexity of $\overline{\mathcal{P}}$-RECOGNITION is essentially the same as that of $\mathcal{P}$-RECOGNITION, so there is a co-additive counterpart to Theorem 6.1.4, but we can do even better by using Theorem 6.1.2. This application illustrates the practical benefits of extending our results in Chapter 4 to indiscompositive properties.

If $\mathcal{Q}$ is additive and $\mathcal{R}$ co-additive, then $\mathcal{Q} \circ \mathcal{R}$ must be NP-hard to recognise if either $\mathcal{Q}$ or $\mathcal{R}$ is NP-hard. By Theorem 6.1.4, it is sufficient for $\mathcal{Q} \neq \mathcal{O}^2$ or $\mathcal{R} \neq \mathcal{K}^2$ to be reducible. We therefore have the following result:

6.1.5. Corollary. *Let $\mathcal{P}$ be in $\mathbb{L}^a_\leq \circ \overline{\mathbb{L}^a_\leq}$. Then $\mathcal{P}$-RECOGNITION is NP-hard except, possibly, if there are efficiently recognisable, irreducible properties $\mathcal{Q} \in \mathbb{L}^a_\leq$ and $\mathcal{R} \in \overline{\mathbb{L}^a_\leq}$ such that $\mathcal{P}$ is $\mathcal{Q}, \mathcal{R}, \mathcal{Q} \circ \mathcal{R}, \mathcal{Q} \circ \mathcal{K}^2, \mathcal{O}^2 \circ \mathcal{R}$ or $\mathcal{O}^2 \circ \mathcal{K}^2$.* $\square$

The family of split graphs, which is just $\mathcal{O} \circ \mathcal{K}$, is known to have a polynomial recognition algorithm; in fact, such an algorithm is known even for $\mathcal{O}^2 \circ \mathcal{K}^2$ [14, 17, 34, Corollary 3] (the algorithm in [15] is incorrect [16]). However, the complexity of the exceptional cases $\mathcal{Q} \circ \mathcal{R}, \mathcal{Q} \circ \mathcal{K}^2$ and $\mathcal{O}^2 \circ \mathcal{R}$ mentioned in Theorem 6.1.5 is still largely unknown.

A result of Alekseev and Lozin [4] leads to a complete classification of the polynomial-time recognisable properties in $\mathbb{L}^a \circ \overline{\mathbb{L}^a}$. For fixed p, let K_p-FREE be the property $\{G \mid K_p \not\leq G\}$. Let p and q be integers, and let $\mathcal{P} \subseteq K_p$-FREE and $\mathcal{Q} \subseteq \overline{K_q}$-FREE be efficiently recognisable induced-hereditary properties (note that $\mathcal{Q}$ cannot be additive). Alekseev and Lozin gave a polynomial-time algorithm for $(\mathcal{P},\mathcal{Q})$-COLOURING, with running time roughly $O(n^{2R(p,q)})$, where $R(p,q)$ is the Ramsey number of p and q.

Now, if $\mathcal{P}$ is hereditary, with completeness $c(\mathcal{P})$, then $\mathcal{P} \subseteq K_{c(\mathcal{P})+1}$-FREE; similarly, $\overline{\mathcal{Q}} \subseteq \overline{K_{c(\mathcal{Q})+1}}$-FREE. Along with Theorem 6.1.2 this gives us:

6.1.6. Theorem [4]). *Let $\mathcal{P}$ and $\mathcal{Q}$ be in $\mathbb{L}^a$. Then $(\mathcal{P}, \overline{\mathcal{Q}})$-RECOGNITION has polynomial-time complexity iff both $\mathcal{P}$-RECOGNITION and $\mathcal{Q}$-RECOGNITION have polynomial-time complexity.* □

Note that, if, for some q, $\mathcal{Q}$ itself is contained in $\overline{K_q}$-FREE, then, by heredity, it cannot contain any graphs with q or more vertices, so it cannot be additive, and $(\mathcal{P}, \mathcal{Q})$-COLOURING would be essentially $\mathcal{P}$-RECOGNITION.

6.2 New directions

A natural problem to tackle next would be classifying the complexity of $\mathcal{R}^k$-RECOGNITION, where $\mathcal{R}$ is neither additive nor co-additive. Certain cases may prove easier to resolve, for example, when $\mathcal{R}$ is irreducible indiscompositive, when $\mathcal{R}$ is hereditary with finitely many forbidden subgraphs, or when there is exactly one connected and one disconnected graph in $\mathcal{F}_{\leq}(\mathcal{R})$.

In the latter category, one of the simplest possible cases is $\mathcal{R} = (\mathcal{O} \cup \mathcal{K})$, where $\mathcal{F}_{\leq}(\mathcal{R}) = \{P_3, \overline{P_3}\}$. Gimbel et al. [38] observed that $G \in \mathcal{O}^k \Leftrightarrow nG \in (\mathcal{O} \cup \mathcal{K})^k$ (where $n = |V(G)|$). Thus, $(\mathcal{O} \cup \mathcal{K})^k$-RECOGNITION is NP-complete for $k \geq 3$ (and, in fact, polynomial for $k = 1, 2$).

Once a problem has been shown to be NP-complete, it is natural to consider whether imposing restrictions on the input graphs leads to an easier problem. For example, GRAPH 4-COLOURING is NP-complete in general, but trivial when restricted to planar graphs. On the other hand, 3-COLOURING is NP-complete even for planar graphs G with maximum degree $\Delta(G) \leq 4$ [36, Thm.s 4.1, 4.2], and for triangle-free graphs with $\Delta(G) \leq 4$ [49]; while k-COLOURING is NP-complete for triangle-free graphs with $\Delta(G) \leq 2^{k+2}$ [49]. Finally, Fiala et al. [35] and, independently, Gimbel showed that P_3-FREE 2-COLOURING is NP-complete for triangle-free planar graphs with $\Delta(G) \leq 4$, while P_3-FREE k-COLOURING is NP-complete on graphs with $\Delta(G) \leq k^2$.

These problems can be phrased as $(\mathcal{D} : \mathcal{P})$-RECOGNITION: given a graph G in the domain $\mathcal{D}$, is G in $\mathcal{P}$? This is just $(\mathcal{D} \cap \mathcal{P})$-RECOGNITION; if $\mathcal{D}$ and $\mathcal{P}$ are both additive induced-hereditary, then so is $\mathcal{D} \cap \mathcal{P}$, with $\mathcal{F}_{\leq}(\mathcal{D} \cap \mathcal{P}) = \min_{\leq}(\mathcal{F}_{\leq}(\mathcal{D}) \cup \mathcal{F}_{\leq}(\mathcal{P}))$. We leave it as an open question to determine when $\mathcal{D} \cap \mathcal{P}$ is reducible (or, equivalently, decomposable).

Some comments about the techniques we used are in order. The most important part of the proof is the 'replicator' gadget. Phelps and Rödl [53, Thm. 6.2] and Brown [22, Thm. 2.3] used different gadgets to perform similar roles. The forcing technique of Theorem 6.1.2 was used previously in [48, Thm. 2] and [18, Lemma 3].

Contacts with Lozin were very helpful, as they spurred the author to look at (K_m-FREE, K_n-FREE)-COLOURING, not knowing it had been settled in [26]. Kratochvíl and Schiermeyer [48] proved a special case of Theorem 6.1.4 that covered the case $m = 2$; I started my proof for general m and n by adapting theirs, and ended up strengthening and simplifying it considerably.

The use of uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partitionable graphs is crucial throughout, but, assuming P $\neq$ NP, it is not sufficient, as such uniquely colourable graphs exist even if the $\mathcal{P}_i$'s are all finite irreducible hereditary compositive properties.

Although we only mention one or two specific forbidden subgraphs explicitly, the proof relies heavily on the fact that *all* forbidden subgraphs are connected; besides, the gadgets we use in Theorem 6.1.4 do depend on $\mathcal{P}$ or $\mathcal{Q}$, as they involve uniquely $(\mathcal{P}, \mathcal{Q})$-colourable graphs.

It is, however, remarkable that we have proved NP-hardness without any particularly detailed knowledge of the forbidden graphs of $\mathcal{P}$ and $\mathcal{Q}$. The determination of $\mathcal{F}_{\leq}(\{\text{line graphs}\})$ and, especially, $\mathcal{F}_{\leq}(\{\text{perfect graphs}\})$, were significant and difficult advances in graph theory, but we did not need them to establish NP-hardness for the recognition of $\{\text{perfect graphs}\} \circ \{\text{line graphs}\}$. This illustrates the advantages of considering additive induced-hereditary properties in such generality.

Chapter 7

New directions and open problems

This short chapter lists some open problems, or areas of research, that we find interesting or significant, or both.

1. Are properties in $\mathbb{L}^c$ (or, at least, in $\mathbb{L}^a$) uniquely factorisable into irreducible properties in $\mathbb{L}$?

2. Similarly, are additive induced-hereditary properties uniquely factorisable into irreducible induced-hereditary properties?

   Equivalently, suppose $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are induced-hereditary, and $\mathcal{P} = \mathcal{Q} \circ \mathcal{R}$. If $\mathcal{P}$ is additive, must $\mathcal{Q}$ and $\mathcal{R}$ be additive too?

3. Are there hereditary properties $\mathcal{P}$ and $\mathcal{Q}$ such that $dc(\mathcal{P} \circ \mathcal{Q}) > dc(\mathcal{P}) + dc(\mathcal{Q})$?

4. For hereditary $\mathcal{P}$, when is the join of n $\mathcal{P}$-maximal graphs $\mathcal{P}^n$-maximal? In particular[1], if $G \in \mathcal{M}^*(\mathcal{P})$, when is the join of n copies of G in $\mathcal{M}^*(\mathcal{P}^n)$?

5. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be irreducible additive hereditary properties. When is the join of n indecomposable $\mathcal{P}_i$-maximal graphs, one from each P_i, a $(P_1 \circ \cdots \circ P_n)$-maximal graph?

[1] Questions 4 and 5 are considered for particular choices of properties in [19]. We could ask, instead, that the join of the n graphs span *some* $\prod \mathcal{P}_i$-maximal graph of decomposability n.

6. Which properties with exactly two forbidden induced-subgraphs are compositive or disjoint compositive? In particular, what can we say about $\mathrm{Forb}_{\leq}(G, H)$, where G is a join, and H is disconnected?

7. Given $\mathcal{F}_{\leq}(\mathcal{P})$ and $\mathcal{F}_{\leq}(\mathcal{Q})$, is there a direct way of getting $\mathcal{F}_{\leq}(\mathcal{P} \circ \mathcal{Q})$?

8. For additive (induced-)hereditary $\mathcal{P}$ and $\mathcal{Q}$, when is $\mathcal{P} \cap \mathcal{Q}$ reducible?

9. Suppose we have an algorithm A that tells us whether a graph G is $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable. Can we use it to produce an actual $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colouring?

   If the $\mathcal{P}_i$'s are hereditary compositive, then the answer is yes. Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$. We can assume the $\mathcal{P}_i$'s are irreducible (this makes no difference to algorithm A), and we have a fixed uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable graph H, which is $\mathcal{P}$-strict and $\mathcal{P}$-maximal. As outlined on p. 95, we can find a $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-colourable graph G_H that contains both G and H. We can then add edges until we get a $\mathcal{P}$-maximal graph G'_H (for each of the $O(|V(G)|^2)$ possible edges, we run algorithm A to check whether we can add the edge while staying in $\mathcal{P}$). This has some $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$-partition $(V_1, \ldots, V_n)$. Suppose some vertex v of G is in V_1; by $\mathcal{P}$-maximality of H, v is joined to every vertex of $V(H) \setminus V_1$, and by $\mathcal{P}$-strictness of H, v is not joined to all vertices of $V(H) \cap V_1$. We can therefore find out easily which vertices of G are in V_1, and similarly for $V_2, \ldots, V_n$.

10. Are Theorems 5.3.2, 5.3.3, 5.3.4 and 6.1.2 true for hereditary or induced-hereditary properties in general?

11. We know that $\mathcal{M}_{\mathcal{B}}(\mathcal{A})_{\subseteq} = \mathcal{M}(\mathcal{A})_{\subseteq}$, and $\mathcal{M}_{\mathcal{B}}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$, but must we have equality? Define

    $$\mathcal{M}^{*\downarrow}_{\mathcal{B}}(\mathcal{A}) := \{G \in \mathcal{A} \mid \exists\, H \in \mathcal{B},\ G + H \in \mathcal{M}^{*\downarrow}(\mathcal{A} \circ \mathcal{B})\}.$$

    It can be checked that $\mathcal{M}^{*\downarrow}_{\mathcal{B}}(\mathcal{A}) \subseteq \mathcal{M}^{*\downarrow}(\mathcal{A})$, but must we have equality?

12. Suppose $\mathcal{P}_1, \ldots, \mathcal{P}_4$ are additive hereditary, $\mathcal{P}_3 \subseteq \mathcal{P}_4$, and $\mathcal{P}_3$ and $\mathcal{P}_4$ do not contain all bipartite graphs. In [12] and [50] it is shown that $\mathcal{P}_1 \circ \mathcal{P}_3 \subseteq \mathcal{P}_2 \circ \mathcal{P}_3$ iff $\mathcal{P}_1 \subseteq \mathcal{P}_2$; and if $\mathcal{P}_1 \circ \mathcal{P}_2 \subseteq \mathcal{P}_3 \circ \mathcal{P}_4$, then either $\mathcal{P}_1 \subseteq \mathcal{P}_3$ and $\mathcal{P}_2 \subseteq \mathcal{P}_4$, or $\mathcal{P}_1 \subseteq \mathcal{P}_4$ and $\mathcal{P}_2 \subseteq \mathcal{P}_3$. How far can the conditions on the $\mathcal{P}_i$'s be relaxed?

The examples given in the proofs of Lemma 5.1.5 and Theorem 5.2.1 show that additivity by itself is not enough. On the other hand, with additivity and other assumptions we can prove more. Recall that $\mathcal{M}^{*\downarrow}(\mathcal{P}) = \{G \in \mathcal{M}(\mathcal{P}) \mid |V(G)| \geq c(\mathcal{P}), dc(G) = dc(\mathcal{P})\}$. Suppose that $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m \subseteq \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_n = \mathcal{Q}$, where all the $\mathcal{P}_i$'s and $\mathcal{Q}_j$'s are irreducible additive hereditary properties. If $\mathcal{P}$ and $\mathcal{Q}$ are "not too far apart", in the sense that every generating set $\mathcal{G} \subseteq \mathcal{M}^{*\downarrow}(\mathcal{P})$ satisfies $\mathcal{G} \cap \mathcal{M}^{*\downarrow}(\mathcal{Q}) \neq \emptyset$, then $m = n$, and essentially the same proof as for Theorem 3.3.2 shows that we can relabel the $\mathcal{Q}_i$'s so that $\mathcal{P}_i \subseteq \mathcal{Q}_i$ for all i.

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