

### Wksh 2, Qn. 4

Let  $G$  be a connected graph with  $n \geq 3$  ( $n$  is the number of vertices). Show that if  $G$  has a cut-edge, then it has a cut-vertex. Show by an example that the converse is false.

**Proof:** Let  $xy$  be a cut-edge. Then  $G - xy$  has 2 components, say  $A$  and  $B$ , with  $x \in A$  and  $y \in B$ .

Since  $n \geq 3$ ,  $A$  and  $B$  cannot contain just one vertex each. Without loss of generality, we consider the case where  $A$  has at least two vertices. So there is some vertex  $v \in A - x$ .

Then  $G - x$  contains both  $v$  and  $y$ . Since there was no  $v - y$  path in  $G - xy$ , there is no  $v - y$  path in  $G - x$ . Thus  $x$  is a cut-vertex.

### Wksh 2, Qn. 6

Let  $G$  be connected. How many components are obtained by removing a  $\lambda$ -set of separating edges? How many components can be obtained by removing a  $\kappa$ -set of separating vertices?

**Proof:** Let  $G$  have  $n$  vertices.

Let  $T$  be a  $\lambda$ -set of separating edges. Let  $xy$  be an edge in  $T$ . For convenience, define  $T' := T - \{xy\}$ , that is,  $T = T' \cup \{xy\}$ .

Let  $G' := (G - T')$ , so  $G - T = G' - xy$ . Note that  $G'$  is connected, while  $G' - xy$  is not, so  $xy$  is a cut-edge of  $G'$ .

Let  $G' - xy$  have components  $C_1, \dots, C_r$ , with  $x$  in  $C_1$  (for example) and  $y \in C_2$ . Now  $(G' - xy) + xy$  is connected, but  $xy$  can only connect  $C_1$  and  $C_2$ , so these must be the only components in  $G' - xy$ .

Thus, removing a  $\lambda$ -set of separating vertices results in just two components.

Let  $S$  be a  $\kappa$ -set of separating vertices. Note that  $G - S$  has  $n - \kappa$  vertices, so we cannot have more than  $n - \kappa$  components. If there are no edges between vertices of  $S$ , then  $G - S$  has  $n - \kappa$  components (with one vertex in each).

[Note that when  $|S| = 1$ , say  $S = \{v\}$ , we cannot have more than  $\deg(v)$  separate components in  $G - S$ .]

### Wksh 2, Qn. 10a

Show that in a simple graph  $G$  if  $\delta \geq n/2$  then  $\lambda = \delta$ .

**Proof:** Let the graph be  $G$ . Let  $T$  be a separating set of edges, that is,  $T$  is the set of edges joining  $A$  to  $V \setminus A$ , for some proper<sup>1</sup> subset  $A$  of  $V(G)$ .

Since we know that  $\lambda \leq \delta$  in any graph, we need to show that  $\lambda \geq \delta$ , that is  $|T| \geq \delta$  for all  $\emptyset \subsetneq A \subsetneq V(G)$ .

For convenience, we define  $a := |A|$ . We can choose  $A$  to be the smaller component of  $G - T$ , so that  $a \leq n/2$ . In particular, we get  $a \leq \delta$ .

Any vertex  $v \in A$  is incident to at least  $\delta$  edges.

At most  $a - 1$  edges join  $v$  to other vertices in  $A$ .

So at least  $\delta - (a - 1)$  edges join  $v$  to vertices in  $V \setminus A$ .

Thus  $|A'| \geq a(\delta - (a - 1))$ . We will show that  $a(\delta - (a - 1)) \geq \delta$ .

We have  $\delta \geq a \Rightarrow \delta(a - 1) \geq a(a - 1) \Rightarrow a\delta - \delta \geq a(a - 1) \Rightarrow a\delta - a(a - 1) \geq \delta$  as claimed.

### Wksh 3, Qn. 1a

Show that if  $G$  is Eulerian, then its line graph  $L(G)$  is Hamiltonian.

**Proof:** Let  $V(G) = \{v_1, \dots, v_n\}$ . If  $G$  is Eulerian, then there is a trail that lists each edge of  $G$  exactly once, say  $v_1v_2, v_1v_7, \dots, v_{19}v_{23}, v_{23}v_1$ , such that each edge is adjacent to the next edge (and the last edge is adjacent to the first one).

This gives us a sequence of vertices of  $L(G)$ , such that each vertex is adjacent to the next one in  $L(G)$  (and the last is adjacent to the first).

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<sup>1</sup>If  $A = \emptyset$ , or  $V \setminus A = \emptyset$ , then removing  $T$  would not disconnect the graph (in fact, we would have  $T = \emptyset$ ). This is why we need  $\emptyset \subsetneq A \subsetneq V(G)$ .

### Wksh 3, Qn. 1b

Show that if  $G$  is Hamiltonian, then its line graph  $L(G)$  is Hamiltonian.

**Proof:** If  $G$  is Hamiltonian, then there is a cycle that contains each vertex exactly once, say  $x_1, x_2, \dots, x_n$ .

Note that  $x_1x_2, x_2x_3, \dots, x_{n-1}x_n$  is a sequence of edges in  $G$  that forms a cycle. Denote this by  $C$ . Note that  $C$  need not contain all edges of  $G$ .

For each  $1 \leq i \leq n$ , let  $E_i$  be the set<sup>2</sup> of edges incident to  $x_i$ . We can therefore traverse all the edges of  $G$  as follows:

Go through each edge in  $E_1$ , then  $x_1x_2$ , then each edge in  $E_2$ , then  $x_2x_3$ , each edge in  $E_3$ , then  $x_3x_4$ , etc.

This sequence does contain all edges of  $G$ , but it may list some edges more than once.

We therefore define  $E'_i := E_i \setminus (C \cup E_1 \cup \dots \cup E_{i-1})$ . The following sequence now lists each edge of  $G$  exactly once: each edge in  $E'_1$ , then  $x_1x_2$ , then each edge in  $E'_2$ , then  $x_2x_3$ , each edge in  $E'_3$ , then  $x_3x_4$ , etc. Moreover, each edge in this sequence is adjacent to the edges before and after it. This therefore gives a cycle in  $L(G)$  that contains each vertex of  $L(G)$  exactly once, i.e. a Hamiltonian cycle in  $L(G)$ .

[Note: To understand this proof, it may help to try it out with an actual graph, e.g.  $K_5$  which has just 10 edges.]

### Wksh 3, Qn. 1a,1b

Show that a graph  $G$  may be neither Eulerian nor Hamiltonian, and yet  $L(G)$  could still be Hamiltonian.

Hint 1: There is an example on 5 vertices or less that will do this. Can you find it?

Hint 2: What is the smallest Hamiltonian graph that you can find? If this is  $L(G)$ , then what could  $G$  be?

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<sup>2</sup>Note that the  $E_i$ 's may overlap.