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The convexity spectra of graphs☆

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Abstract

Let D be a connected oriented graph. A set $S \subseteq V(D)$ is *convex* in D if, for every pair of vertices $x, y \in S$, the vertex set of every $x - y$ geodesic ($x - y$ shortest dipath) and $y - x$ geodesic in D is contained in S. The *convexity number* $\text{con}(D)$ of a nontrivial oriented graph D is the maximum cardinality of a proper convex set of D. Let G be a graph. We define that $S_C(G) = \{\text{con}(D)$: D is an orientation of $G\}$ and $S_{SC}(G) = \{\text{con}(D)$: D is a strongly connected orientation of $G\}$. In the paper, we show that, for any $n \geqslant 4$, $1 \leqslant a \leqslant n-2$, and $a \neq 2$, there exists a 2-connected graph G with n vertices such that $S_C(G) = S_{SC}(G) = \{a, n-1\}$ and there is no connected graph G of order $n \geqslant 3$ with $S_{SC}(G) = \{n-1\}$. Then, we determine that $S_C(K_3) = \{1, 2\}$, $S_C(K_4) = \{1, 3\}$, $S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$, $S_C(K_5) = \{1, 3, 4\}$, $S_C(K_6) = \{1, 3, 4, 5\}$, $S_{SC}(K_5) = S_{SC}(K_6) = \{1, 3\}$, $S_C(K_n) = \{1, 3, 5, 6, \ldots, n-1\}$, $S_{SC}(K_n) = \{1, 3, 5, 6, \ldots, n-2\}$ for $n \geqslant 7$. Finally, we prove that, for any integers n, m, and k with $n \geqslant 5$, $n+1 \leqslant m \leqslant \binom{n}{2} - 1$, $1 \leqslant k \leqslant n-1$, and $k \neq 2, 4$, there exists a strongly connected oriented graph D with n vertices, m edges, and convexity number k.

MSC: 05C12; 05C20; 05C35

Keywords: Convexity number; Convex set; Spectrum; Oriented graph

1. Introduction

Convexity in graphs is discussed in the book by Buckley and Harary [1] and studied by Harary and Nieminen [5]. The concept of convexity number of an oriented graph was first introduced by Chartrand et al. [3].

Graphs considered in the paper are finite, without loops or multiple edges. In a graph $G = (V, E)$, V(or $V(G)$) and E(or $E(G)$) denote the vertex set and the edge set of G, respectively. A *cut vertex* v is a vertex in a connected graph G with $G - \{v\}$ being disconnected. A *block* of a graph G is a maximal connected subgraph of G without a cut vertex. A block B of G is an *end block* of a graph G if B contains exactly one cut vertex of G. An *oriented graph* is an orientation of some graph. In an oriented graph $D = (V, E)$, V(or $V(D)$) and E(or $E(D)$) denote the vertex set and the edge set of D, respectively. An *oriented subgraph* $D' = (V', E')$ of an oriented graph $D = (V, E)$ is an oriented graph with $V' \subseteq V$ and $E' \subseteq E$. An oriented graph is *connected* if its underlying graph is connected. A *dipath* is a sequence $(v_1, v_2, \ldots, v_k)$ of vertices of an oriented graph D such that $v_1, v_2, \ldots, v_k$ are distinct and $(v_i, v_{i+1}) \in E(D)$ for $i = 1, 2, \ldots, k-1$. An oriented graph is called *strongly connected* if for any two distinct vertices u and v, there exists

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a dipath from u to v. A *strong component* of an oriented graph D is a maximal strongly connected oriented subgraph in D.

A $u-v$ *geodesic* in a digraph D is a shortest $u-v$ dipath and its length is $d_D(u,v)$. The *closed interval* $I[u,v]$ between two vertices u and v of a digraph D is the set of all vertices lying on a $u-v$ or $v-u$ geodesic(if it exists) in D. If there is no $u-v$ and $v-u$ geodesics, then we define that $I[u,v]_D=\{u,v\}$. A nonempty subset S of the vertex set of a digraph D is called a *convex set* of D if, for every $u,v\in S$, every vertex lying on a $u-v$ or $v-u$ geodesic belongs to S. For a nonempty subset A of $V(D)$, the *convex hull* $[A]$ is the minimal convex set containing A. Thus $[S]=S$ if and only if S is convex in D. The *convexity number* $\mathrm{con}(D)$ of a digraph D is the maximum cardinality of a proper convex set of D. A *maximum convex set* S of a digraph D is a convex set with cardinality $\mathrm{con}(D)$. Since every singleton vertex set is convex in a connected oriented graph D, $1\leqslant \mathrm{con}(D)\leqslant n-1$. The degree $deg(v)$ of a vertex v in an oriented graph is the sum of its indegree and outdegree; that is, $deg(v)=id(v)+od(v)$. A vertex v is an *end-vertex* if $deg(v)=1$. A *source* is a vertex having positive outdegree and indegree 0, while a *sink* is a vertex having positive indegree and outdegree 0. For a vertex v of D, let $N^+(v)=\{x\colon (v,x)\in E(D)\}$ and $N^-(v)=\{x\colon (x,v)\in E(D)\}$. So if v is a source, then $N^-(v)=\emptyset$, while if v is a sink, then $N^+(v)=\emptyset$. A vertex v of D is a *transitive vertex* if $od(v)>0$, $id(v)>0$, and for every $u\in N^+(v)$ and $w\in N^-(v)$, $(w,u)\in E(D)$. For a nontrivial connected graph G, we define that the *convexity-spectrum* $S_C(G)$ of a graph G as the set of convexity numbers of all orientations of G and the *strong convexity-spectrum* $S_{SC}(G)$ of a graph G as the set of convexity numbers of all strongly connected orientations of G. If G has no strongly connected orientation, then $S_{SC}(G)$ is empty. Then the *lower orientable convexity number* $\mathrm{con}^-(G)$ of G is the minimum convexity number among the orientations of G and the *upper orientable convexity number* $\mathrm{con}^+(G)$ is the maximum convexity number among the orientations of G; that is, $\mathrm{con}^-(G)=\min S_C(G)$ and $\mathrm{con}^+(G)=\max S_C(G)$. Hence, for every nontrivial connected graph G of order n, $1\leqslant \mathrm{con}^-(G)\leqslant \mathrm{con}^+(G)\leqslant n-1$.

Chartrand et al. [3] characterized the nontrivial connected oriented graphs of order n with convexity number $n-1$, and showed that there is no connected oriented graph of order at least 4 with convexity number 2. They also showed that every pair k,n of positive integers with $1\leqslant k\leqslant n-1$ and $k\neq 2$ is realizable as the convexity number and order, respectively, of some connected oriented graph.

In the paper, we show that for any $n\geqslant 4$, $1\leqslant a\leqslant n-2$ and $a\neq 2$, there exists a 2-connected graph G with n vertices such that $S_C(G)=S_{SC}(G)=\{a,n-1\}$, and there is no connected graph G of order $n\geqslant 3$ with $S_{SC}(G)=\{n-1\}$. Then we prove that $S_C(K_3)=\{1,2\}$, $S_C(K_4)=\{1,3\}$, $S_{SC}(K_3)=S_{SC}(K_4)=\{1\}$, $S_C(K_5)=\{1,3,4\}$, $S_C(K_6)=\{1,3,4,5\}$, $S_{SC}(K_5)=S_{SC}(K_6)=\{1,3\}$, $S_C(K_n)=\{1,3,5,6,\ldots,n-1\}$, $S_{SC}(K_n)=\{1,3,5,6,\ldots,n-2\}$ for $n\geqslant 7$. Finally, for any integers n, m, and k with $n\geqslant 5$, $n+1\leqslant m\leqslant \binom{n}{2}-1$, $1\leqslant k\leqslant n-1$, and $k\neq 2,4$, we prove that there exists a strongly connected oriented graph D with n vertices, m edges, and convexity number k.

2. Constructing oriented graphs with fixed lower orientable convexity number and upper orientable convexity number

For each connected graph G of order $n\geqslant 2$, there exists an acyclic orientation D of G. Then D has a source v and $V(D)-\{v\}$ is a convex set. This implies that $n-1\in S_C(G)$. The following two useful results were proved by Chartrand et al. in [3].

Theorem 1 (*Chartrand et al., [3]*). *Let D be a connected oriented graph of order* $n\geqslant 2$. *Then* $\mathrm{con}(D)=n-1$ *if and only if D contains a source, sink, or transitive vertex.*

Theorem 2 (*Chartrand et al., [3]*). *There is no connected oriented graph of order at least* 4 *with convexity number* 2.

Farrugia [4] proved that a connected graph of order at least 3 has no end-vertex if and only if $\mathrm{con}^-(G)$ and $\mathrm{con}^+(G)$ are different.

Theorem 3 (*Farrugia [4]*). *Suppose G is a connected graph of order* $n\geqslant 3$. *Then* $\mathrm{con}^-(G)<\mathrm{con}^+(G)$ *if and only f G has no end-vertex.*

The following result is immediate from Theorem 3.

Corollary 4. *Suppose G is a connected graph of order* $n \geqslant 3$. *Then* $|S_C(G)| \geqslant 2$ *if and only if G has no end-vertex.*

By Theorem 4, for any connected graph G of order $n \geqslant 3$, $|S_C(G)| = 1$ if and only if G has an end-vertex.

If a connected graph G has a cut vertex then there is a lower bound of $\mathrm{con}^-(G)$ related to the cardinality of a minimum end block of G.

Lemma 5. *Let G be a nontrivial connected graph of order* $n \geqslant 3$ *and B be a minimum end block B. If D is an orientation of G, then* $\mathrm{con}(D) \geqslant n - |B| + 1$.

Proof. Suppose D is an orientation of G and u is the cut vertex of G with $u \in V(B)$. Let $D - \{u\} = D_1 \cup D_2 \cup \cdots \cup D_k$ where each D_i is a component of $D - \{u\}$ for $i = 1, 2, \ldots, k$. Without loss of generality, $V(D_1) = V(B) - \{u\}$. It is clear that $V(D) - V(D_1)$ is a convex set and $|V(D) - V(D_1)| = n - |B| + 1$. Then $\mathrm{con}(D) \geqslant n - |B| + 1$. □

According to Lemma 5 and B being a minimum end block of G, we have that

Theorem 6. *For positive integer* $n \geqslant 3$, *there exists a connected graph G with* $\mathrm{con}^-(G) \geqslant (n + 1)/2$.

Next, we show that for any $1 \leqslant a \leqslant n - 2$ with $a \neq 2$, there exists a 2-connected graph G with n vertices such that $\mathrm{con}^-(G) = a$ and $\mathrm{con}^+(G) = n - 1$.

Theorem 7. *For every pair of positive integers n and a with* $n \geqslant 4$, $1 \leqslant a \leqslant n - 2$ *and* $a \neq 2$, *there exists a 2-connected graph G with n vertices such that* $S_C(G) = S_{SC}(G) = \{a, n - 1\}$.

Proof. For $a = 1$, define a connected graph $G_1 = (V_1, E_1)$ with $V_1 = \{u, v, v_1, v_2, \ldots, v_{n-2}\}$ and $E_1 = \{uv\} \cup \{uv_i, v_i v: i = 1, 2, \ldots, n - 2\}$. Then G_1 is 2-connected. Let D be an orientation of G_1. Without loss of generality, $(u, v) \in E(D)$. If D has a source, sink, or transitive vertex then $\mathrm{con}(D) = n - 1$. So, suppose that D has no source, sink, or transitive vertex. Then D is strongly connected. Hence there exists i such that (v, v_i, u) is a geodesic in D. If, for every $1 \leqslant j \leqslant n - 2$, (v, v_j, u) is a geodesic in D then, for any two distinct vertices x and y of D, $u, v \in I[x, y]$ and $[\{x, y\}] = V(D)$; that is, $\mathrm{con}(D) = 1$. Hence we have that $S_C(G_1) = S_{SC}(G_1) = \{1, n - 1\}$.

Assume that $3 \leqslant a \leqslant n-2$ and define G_a as the graph with $V(G_a) = \{u, v, u_1, \ldots, u_{n-a}, v_1, \ldots, v_{a-2}\}$ and $E(G_a) = \{uv, uu_1, vu_{n-a}\} \cup \{uv_i, vv_i: 1 \leqslant i \leqslant a - 2\} \cup \{u_i u_{i+1}: 1 \leqslant i \leqslant n - a - 1\}$. It is evident that G_a is 2-connected.

Let D_{n-1} be the orientation of G_a with $E(D_{n-1}) = \{(u, v), (u, v_1), (v_1, v), (v, u_{n-a}), (u_1, u)\} \cup \{(v, v_i), (v_i, u): 2 \leqslant i \leqslant a - 2\} \cup \{(u_{i+1}, u_i): 1 \leqslant i \leqslant n - a - 1\}$. For $a \geqslant 3$, D_{n-1} is a strongly connected graph and v_1 is a transitive in D_1. Then $\mathrm{con}(D_{n-1}) = n - 1$; that is $n - 1 \in S_C(G_a) \cap S_{SC}(G_a)$.

Let D be an orientation of G_a with $\mathrm{con}(D) < n - 1$. By Theorem 1, D has no sink, source, or transitive vertex. Without loss of generality, assume that $(u, v) \in E(D)$. Then (v, v_i, u) is a geodesic in D for $i = 1, 2, \ldots, a - 2$. By $n - a \geqslant 2$, the length of the path $(u, u_1, \ldots, u_{n-a}, v)$ in G_a is greater than 2. Since $u_1, u_2, \ldots, u_{n-a}$ are not sources or sinks, either $(u, u_1, u_2, \ldots, u_{n-a}, v)$ or $(v, u_{n-a}, u_{n-a-1}, \ldots, u_1, u)$ is in D. For either $(u, u_1, u_2, \ldots, u_{n-a}, v)$ or $(v, u_{n-a}, u_{n-a-1}, \ldots, u_1, u)$ being in D, D is strongly connected and the set $\{u, v_1, \ldots, v_{a-2}, v\}$ is a proper convex set in D. If a convex set S contains vertices u_i and x for some $x \in V(D) - \{u_i\}$ then $I[u_i, x]$ contains vertices u, v, $u_1, \ldots, u_{n-a}$. This implies that $[\{u_i, x\}] = V(D)$. So, if S is a proper convex set in D then S does not contain vertices u_j. Thus $\{u, v_1, \ldots, v_{a-2}, v\}$ is the unique maximum proper convex set of D. Hence $\mathrm{con}(D) = a$. The proof is complete. □

Corollary 8. *For every pair of positive integers n and a with* $n \geqslant 4$, $1 \leqslant a \leqslant n - 1$ *and* $a \neq 2$, *there exists a connected graph G with n vertices such that* $S_C(G) = \{a, n - 1\}$.

Proof. For $1 \leqslant a \leqslant n - 2$ and $a \neq 2$, by Theorem 7, there is a connected graph G such that $S_C(G) = \{a, n - 1\}$. If $a = n - 1$, then we take G to be a tree; $S_C(G) = \{n - 1\}$. □

Corollary 9. *For every pair of positive integers n and a with* $n \geqslant 4$, $1 \leqslant a \leqslant n - 1$ *and* $a \neq 2$, *there exists a connected graph G with n vertices such that* $\mathrm{con}^-(G) = a$ *and* $\mathrm{con}^+(G) = n - 1$.

Theorem 10 (*Menger's Theorem [7]*). *Suppose n and k are positive integers with $n \geqslant k+1$. Then a graph G of order n is k-connected if and only if any two distinct vertices of G are connected by at least k internally-disjoint paths.*

The following theorem can be proved by Theorem 2 of [4]. We give an another proof in the following.

Theorem 11. *There is no connected graph G of order $n \geqslant 3$ with $S_{SC}(G) = \{n-1\}$.*

Proof. Suppose G is a connected graph with $S_{SC}(G) = \{n-1\}$. Then there exists a strongly connected orientation of G. This implies that every block of G is 2-connected; that is, each block has at least 3 vertices. If every block of G has a strongly connected orientation without transitive vertex then there is a strongly connected orientation D without source, sink, transitive vertex. By Theorem 1, $\mathrm{con}(D) < n-1$. This contradicts our assumption that $S_{SC}(G) = \{n-1\}$. So, it remains to construct a strongly connected orientation of G without transitive vertex in the next paragraph.

Suppose G' is 2-connected. Claim that there exists a strongly connected orientation D of G' without a transitive vertex. By Menger's Theorem, we have that the property (*): for three distinct vertices x, y, z of G', there exist two paths P_1 from x to y and P_2 from x to z in G' such that $V(P_1) \cap V(P_2) = \{x\}$.

Since G' is 2-connected, there exists a cycle $(v_1, v_2, \ldots, v_k, v_1)$ which is an induced subgraph of G'. (i.e. there is not any chord in $(v_1, v_2, \ldots, v_k, v_1)$). Define the directed cycle $(v_1, v_2, \ldots, v_k, v_1)$ in D. We have that, for each i, either v_i has an out neighbor and an in neighbor which are nonadjacent or (v_1, v_2, v_3, v_1) is a directed cycle in D. This implies that vertices $v_1, v_2, \ldots, v_k$ are not source, sink, or transitive vertex. Let $S_1 = \{v_1, v_2, \ldots, v_k\}$. If $V(G') - S_1$ is not empty then, by the property (*), there is a shortest path $(v_i, x_1, \ldots, x_r, v_j)$ of G' satisfying ($k \geqslant i > j \geqslant 1$ and $(i, j) \neq (k, 1)$) or $(i, j) = (1, k)$ such that $r \geqslant 1$ and $x_1, \ldots, x_r \notin S_1$. Define the directed path $(v_i, x_1, \ldots, x_r, v_j)$ is in D. We observe that if $r > 1$ then each vertices of $x_1, \ldots, x_r$ has an out neighbor and an in neighbor that are nonadjacent in G' by $(v_i, x_1, \ldots, x_r, v_j)$ being shortest. If $r = 1$ then either $v_i v_j \notin E(G')$or $(v_j, v_i) \in E(D)$. Let $S_2 = S_1 \cup \{x_1, \ldots, x_r\}$. The other edges between two vertices of S_2 are assigned random directions in D. Therefore each vertex of $x_1, \ldots, x_r$ is not source, sink, or transitive vertex in D. Repeatedly, we can find a shortest path $(x, y_1, \ldots, y_s, y)$ with $xy \notin E(G')$ or $(y, x) \in E(D)$ such that $s \geqslant 1$ and $y_1, \ldots, y_s \notin S_i$, then define the directed path $(x, y_1, \ldots, y_s, y)$ is in D, and let $S_{i+1} = S_i \cup \{y_1, \ldots, y_s\}$. The other edges between two vertices of S_{i+1} are assigned random directions in D. Until $S_{i+1} = V(G')$, we obtained a strongly connected orientation D of G without source, sink, or transitive vertex. □

3. Strong convexity spectra of complete graphs

In this section, we determine the convexity-spectra and the strong convexity-spectra of complete graphs. Since $S_C(G) = \{\mathrm{con}(D)\colon D$ is an orientation of $G\}$ and $S_{SC}(G) = \{\mathrm{con}(D)\colon D$ is a strong orientation of $G\}$, $S_{SC}(G) \subseteq S_C(G)$. An orientation of the complete graph of order n is called a *tournament* of order n. In a strongly connected graph D, the diameter of D is denoted by $diam(D)$.

We find some strongly connected tournaments D of order $n \geqslant 3$ with $\mathrm{con}(D) = 1$.

Lemma 12. *Suppose n is a positive integer with $n \geqslant 3$ and $n \neq 4$. Then there exists a strongly connected tournament D of order n with $d(D) = 2$ and $\mathrm{con}(D) = 1$ and every strongly connected tournament of order 4 has diameter 3.*

Proof. If $n = 4$, then all strongly connected tournaments of order 4 are isomorphic. It is easy to check that the diameter of every strongly connected tournament of order 4 is 3.

Suppose n is a positive integer with $n \geqslant 3$ and $n \neq 4$. For $n = 3$, a directed cycle D_1 with the vertex set $\{a, b, c\}$ has $d(D_1) = 2$ and $\mathrm{con}(D_1) = 1$. For $n = 5$, Let D_2 be the oriented graph with $V(D_2) = V(D_1) \cup \{x, y\}$ and $E(D_2) = E(D_1) \cup \{(u, x), (y, u)\colon u \in V(D_1)\} \cup \{(x, y)\}$. We can find that D_2 has $d(D_2) = 2$ and $\mathrm{con}(D_2) = 1$. For $n = 6$, let D_3 be the oriented graph with $V(D_3) = V(D_2) \cup \{z\}$ and $E(D_3) = E(D_2) \cup \{(a, z), (b, z), (x, z), (z, c), (z, y)\}$. We also can find that D_3 has $d(D_3) = 2$ and $\mathrm{con}(D_3) = 1$. For $n \geqslant 7$, if we have a strongly connected tournament D' with order $n-1$, $d(D') = 2$, and $\mathrm{con}(D') = 1$, then let D be the oriented strongly connected tournament with the vertex set $V(D') \cup \{s, t\}$ and the edge set $E(D') \cup \{(u, s), (t, u)\colon u \in V(D')\} \cup \{(s, t)\}$.We have that D has $d(D) = 2$ and $\mathrm{con}(D) = 1$. By induction, we can get the theorem. □

Lemma 13. *If D is a strongly connected tournament of order $n \geqslant 3$, then $\mathrm{con}(D) \leqslant n-2$.*

Proof. Let S be a set of vertices in D with $|S| = n - 1$. Then there exists a vertex $v \in V(D) - S$. Let $X = \{x \in V(D): (x, v) \in E(D)\}$ and $Y = \{y \in V(D): (v, y) \in E(D)\}$. By D being a strongly connected tournament, there exist $x \in X$ and $y \in Y$ such that $(y, x) \in E(D)$. Thus (x, v, y) is a geodesic in D. Then S is not a convex set; that is, $\mathrm{con}(D) \leqslant n - 2$. □

Lemma 14. *For $n \geqslant 6$ being a positive integer, $4 \notin S_{SC}(K_n)$.*

Proof. Suppose D be a strongly tournament with S being a proper convex set of 4 vertices in D. Then the induced subgraph H of S is strongly connected. Since $diam(H) = 3$, there is a geodsic (a, b, c, d) in the induced subgraph of S. Let $A = \{u: u \in V(D) - S$ and $(u, a) \in E(D)\}$ and $B = \{v: v \in V(D) - S$ and $(d, v) \in E(D)\}$. Take $u \in A$ and $v \in B$. Since (a, b, c, d) is a dipath of D and S is convex, $(u, b), (c, v) \in E(D)$. If $(u, b), (c, v) \in E(D)$ then, by S being convex, $(u, c), (b, v) \in E(D)$. Similarly, we have that $(u, d), (a, v) \in E(D)$. By $diam_D(a, d) = 3$ and S being convex, A and B are disjoint and $(u, v) \in E(D)$ for $u \in A$ and $v \in B$. If $V(D) - (S \cup A \cup B)$ is empty, then D is not strongly connected. So, there exists a vertex $x \in V(D) - (S \cup A \cup B)$ with $(a, x), (x, d) \in E(D)$. It contradicts that $diam_D(a, d) = 3$. Hence, $4 \notin S_{SC}(K_n)$ for $n \geqslant 6$. □

We combine the ideas in 12–14 to get the following Theorem.

Theorem 15. *$S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$, $S_{SC}(K_5) = S_{SC}(K_6) = \{1, 3\}$, and $S_{SC}(K_n) = \{1, 3, 5, 6, \ldots, n - 2\}$ for integer $n \geqslant 7$.*

Proof. For $n = 3$ or 4, by Theorem 2, Lemma 12 and 13, $S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$. For $n \geqslant 5$, by Lemma 12, $1 \in S_{SC}(K_n)$; and by Theorem 2 and Lemma 13 and 14, $2, 4, n - 1 \notin S_{SC}(K_n)$. In the following paragraphs, we construct a strongly connected tournament D with order $n \geqslant 5$ and $\mathrm{con}(D) = k$ for $3 \leqslant k \leqslant n - 2$ and $k \neq 4$.

Suppose $n \geqslant 5$, $3 \leqslant k \leqslant n - 2$ and $k \neq 4$. Let the vertex set of K_n be $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_{n-k}\}$, $V = \{v_1, v_2, \ldots, v_k\}$, and $U = \{u_1, u_2, \ldots, u_{n-k}\}$.

By Lemma 12, there exists a strongly connected oriented graph D_1 with the vertex set V, $(v_k, v_1) \in E(D_1)$, $d(D_1)=2$, and $\mathrm{con}(D_1) = 1$. Let D_2 be the oriented graph with the vertex set U and the edge set $\{(u_i, u_j): 1 \leqslant i < j \leqslant n - k\} - \{(u_1, u_{n-k})\} \cup \{(u_{n-k}, u_1)\}$ for $n - k \geqslant 3$. If $n - k = 2$ then the edge set of D_2 is defined by $\{(u_2, u_1)\}$. Let D_3 be the oriented graph with the vertex set $V \cup U$ and the edge set $\{(u_1, v_i): 1 \leqslant i \leqslant k\} \cup \{(v_i, u_l): 1 \leqslant i \leqslant k$ and $2 \leqslant l \leqslant n - k\}$. Define that D is a strongly connected orientation of K_n with $E(D) = E(D_1) \cup E(D_2) \cup E(D_3)$.

By Lemma 12, $d_D(v_i, v_j) \leqslant 2$ for all $1 \leqslant i < j \leqslant k$. And $u_l \notin I[v_i, v_j]$ in D for all i, j, l. Then V is a convex set of D.

Let S be a convex set of D. If there exist $1 \leqslant l < m \leqslant n - k$ such that $u_l, u_m \in S$, then $I[u_m, u_l]$ contains vertices u_1 and u_{n-k}. Since (u_1, u_p, u_{n-k}) and (u_1, v_i, u_{n-k}) are $u_1 - u_{n-k}$ geodesics in $V(D)$ for all $1 < p < n-k$ and $1 \leqslant i \leqslant k$, $S = V(D)$. If there exist $1 \leqslant l \leqslant n - k$ and $1 \leqslant m \leqslant k$ such that $u_l, v_m \in S$, then $I[u_l, v_m]$ contains vertices u_1 and u_{n-k}. By the same above reason, $S = V(D)$. So, we have that V is a maximum convex set with $V \neq V(D)$. Thus, $\mathrm{con}(D) = |V| = k$. □

Lemma 16. *For positive integer $n \geqslant 7$, $4 \notin S_C(K_n)$.*

Proof. Suppose D is a tournament of order $n \geqslant 7$. If D is strongly connected then, by Lemma 14, $\mathrm{con}(D) \neq 4$. Assume that D is not strongly connected. Then there exists a strong component S of D such that $(x, y) \in E(D)$ for each $x \in V(D) - S$ and $y \in S$. If $|S| = 1$ or $|V(D) - S| = 1$, then D has a sink or a source; that is, $\mathrm{con}(D) = n - 1 \neq 4$. If $|S|, |V(D) - S| > 1$, then, for $x \in V(D) - S$ and $y \in S$, $S \cup \{x\}$ and $(V(D) - S) \cup \{y\}$ are proper convex sets in D; that is, $\mathrm{con}(D) \geqslant n/2 + 1 > 4$. □

Theorem 17. *$S_C(K_3) = \{1, 2\}$, $S_C(K_4) = \{1, 3\}$, $S_C(K_5) = \{1, 3, 4\}$, $S_C(K_6) = \{1, 3, 4, 5\}$ and $S_C(K_n) = \{1, 3, 5, 6, \ldots, n - 1\}$ for integer $n \geqslant 7$.*

Proof. Since every acyclic orientation D has a source, $\mathrm{con}(D) = n - 1$. Then $n - 1 \in S_C(K_n)$ for $n \geqslant 2$. By $S_{SC}(G) \subseteq S_C(G)$, Theorem 15, and Lemma 16, we have that $S_C(K_3)=\{1, 2\}$, $S_C(K_4)=\{1, 3\}$, $S_C(K_5)=\{1, 3, 4\}$, and $S_C(K_n)=\{1, 3, 5, 6, \ldots, n-1\}$ for integer $n \geqslant 7$. For K_6, we have an orientation D with $V(D)=\{v_1, v_2, v_3, v_4, v_5, v_6\}$

and $E(D) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_4, v_5), (v_5, v_6), (v_6, v_4)\} \cup \{(v_i, v_j): 1 \leqslant i \leqslant 3 \text{ and } 4 \leqslant j \leqslant 6\}$ with $\text{con}(D) = 4(\{v_1, v_2, v_3, v_4\}$ is a maximum convex set). Thus, $S_C(K_6) = \{1, 3, 4, 5\}$. □

4. Constructing strongly connected oriented graphs with fixed order, size, and convexity number

Analyzing the orientation D in the proof of Theorem 15, we have the following remarks. (We use the same notations in the following remarks.)

Remark 18. For $1 \leqslant i < j \leqslant k$, $\{v_1, v_k\} \subseteq [\{v_i, v_j\}]$ and $[\{v_1, v_k\}] = V$. Then V is a convex set in D.

Remark 19. (1) For $2 \leqslant l \leqslant n-k-1$, (u_1, u_l, u_{n-k}) is a geodesic in D.

(2) For $1 \leqslant i \leqslant k$, if k is odd then (u_1, v_i, u_{n-k}) is a geodesic in D, and if k is even and $i \neq k-1$ then (u_1, v_i, u_{n-k}) is a geodesic in D and (v_k, v_{k-1}, v_1) is a geodesic in D.

Remark 20. Every vertex of U except u_1 and u_{n-k} belongs to a unique $u_1 - u_{n-k}$ geodesic in D.

Remark 21. For $1 \leqslant l < m \leqslant n-k$, $\{u_1, u_{n-k}\} \subseteq [\{u_l, u_m\}]$ and $[\{u_1, u_{n-k}\}] = V(D)$. Then $[\{u_l, u_m\}] = V(D)$.

Remark 22. For $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant n-k$, $\{u_1, u_{n-k}\} \subseteq [\{v_i, u_l\}]$. Then $[\{v_i, u_l\}] = V(D)$.

Lemma 23. *Let D be a connected oriented graph of order $n \geqslant 5$. If $V(D) = V \cup U$ where $V = \{v_1, v_2, \ldots, v_k\}$, $U = \{u_1, u_2, \ldots, u_{n-k}\}$, $k \geqslant 3$ and $n-k \geqslant 2$ satisfying two conditions*:

(1) $[\{v_i, v_j\}] = V$, *for every $i < j$ and*
(2) $[\{u_1, u_{n-k}\}] = V(D)$ *and $u_1, u_{n-k} \in [\{x, y\}]$ for every two vertices x, y with $x \in U$ and $y \in V(D) - \{x\}$,*

then V is the unique maximum convex set in D and $\text{con}(D) = |V|$.

Proof. By condition (1), we have that V is a convex set in D. According to the condition (2), the convex hull of every pair of vertices in U is $V(D)$ and the convex hull of every pair u, v with $u \in U,\ v \in V$ is also $V(D)$. Then we have V is the unique maximum convex set in D and $\text{con}(D) = |V|$. □

By Theorem 2, there is no connected graph G of order $n \geqslant 4$ with $2 \in S_{SC}(G)$. In the first theorem of this section, we consider the existence of strongly connected oriented graphs of order n, size m, and convexity number k where $n \geqslant 5$, $n+1 \leqslant m \leqslant \binom{n}{2}$, and $3 \leqslant k \leqslant n-2$.

Theorem 24. *For integers k, n, m with $n \geqslant 5$, $3 \leqslant k \leqslant n-2$, $k \neq 4$, and $n+1 \leqslant m \leqslant \binom{n}{2}$, there exists a strongly connected oriented graph with n vertices, m edges, and convexity number k.*

Proof. Let $n,\ m$ and k be positive integers with $n \geqslant 5$, $3 \leqslant k \leqslant n-2$, $n+1 \leqslant m \leqslant \binom{n}{2}$. In the following, we construct a strongly connected oriented graph with n vertices, m edges, and convexity number k by examining different cases.

(a) First, for $m = \binom{n}{2}$, by Theorem 15, there exists a strongly connected tournament D of order n and convexity number k.
(b) For $\binom{n}{2} - k(n-k) + 4 \leqslant m < \binom{n}{2}$, there is a strongly connected oriented graph D with $E(D) = E(D_1) \cup E(D_2) \cup E(D_3)$ in the proof of Theorem 15. Let $S \subseteq E(D_3) - \{(u_1, v_1), (u_1, v_k), (v_1, u_{n-k}), (v_k, u_{n-k})\}$ with $|S| = \binom{n}{2} - m$ and $H_m = D - S$. By V being a convex set of D and $S \subseteq E(D_3)$, V is still a convex set in H_m. By Lemma 12, for every $i < j$, $[\{v_i, v_j\}]_{H_m} = V$. Since (u_1, u_i, u_{n-k}), (u_1, v_1, u_{n-k}), (u_1, v_k, u_{n-k}) are geodesics in H_m, $[\{u_1, u_{n-k}\}]_{H_m} = V(H_m)$. And, for $i < j$, $u_1, u_{n-k} \in I_{H_m}[u_j, u_i]$. This implies that $[\{u_i, u_j\}]_{H_m} = V(H_m)$. By Lemma 23, V is the unique maximum convex set in H_m and $\text{con}(H_m) = |V| = k$.
(c) For $\binom{n}{2} - k(n-k) + 4 - \binom{n-k-2}{2} \leqslant m < \binom{n}{2} - k(n-k) + 4$. Let $D' = D - E(D_3) \cup \{(u_1, v_1), (u_1, v_k), (v_1, u_{n-k}), (v_k, u_{n-k})\}$, $S \subseteq \{(u_i, u_j): 2 \leqslant i < j \leqslant n-k-1\}$ with $|S| = \binom{n}{2} - k(n-k) + 4 - m$, and

$H_m = D' - S$. Similar to part (b), by Theorem 23, V is the unique maximum convex set in H_m and $\mathrm{con}(H_m) = |V| = k$.

(d) For $2n-2 \leqslant m < \binom{n}{2} - k(n-k) + 4 - \binom{n-k-2}{2}$. Define D' to be a strongly connected oriented graph with $V(D') = V \cup U$ and $E(D') = E_1 \cup E_2 \cup \{(u_1, v_1), (u_1, v_k), (v_1, u_{n-k}), (v_k, u_{n-k})\}$ where $E_1 = \{(v_j, v_i): i < j\} - \{(v_k, v_1)\} \cup \{(v_1, v_k)\}$ and $E_2 = \{(u_{n-k}, u_1)\} \cup \{(u_1, u_i), (u_i, u_{n-k}): 1 < i < n-k\}$. Then $|E(D')| = \binom{n}{2} - k(n-k) + 4 - \binom{n-k-2}{2} = \binom{k}{2} + 4 + 2(n-k-2) + 1$. Let $S \subseteq \{(v_j, v_i): 2 \leqslant i < j \leqslant k-1\}$ with $|S| = \binom{k}{2} + 4 + 2(n-k-2) + 1 - m$ and $H_m = D' - S$. We have that $|E(H_m)| = m$, and V is convex in H_m. For $x \in U$ and $y \in V(H_m) - \{x\}$, $u_1, u_{n-k} \in I_{H_m}[x, y]$, $v_1, v_k, u_1, u_2, \ldots, u_{n-k} \in I_{H_m}[u_1, u_{n-k}]$, and $I_{H_m}[v_1, v_k] = V$. By Lemma 23, V is the unique maximum convex set in H_m and $\mathrm{con}(H_m) = |V| = k$.

(e) For $m = 2n-3$. (i) If $k = n-2$ then let $D = (V, E)$ be the digraph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_1, v_i), (v_i, v_{n-2}): i = 2, \ldots, n-3\} \cup \{(v_{n-2}, v_1), (v_{n-2}, v_{n-1}), (v_{n-1}, v_n), (v_n, v_1), (v_1, v_{n-1})\}$. Then $|E| = 2(n-4) + 5 = 2n-3$ and $\{v_1, v_2, \ldots, v_{n-2}\}$ is a convex set. If S is a convex set with $|S| \geqslant n-1$ then there exist $2 \leqslant i \leqslant n-2$ and $n-1 \leqslant j \leqslant n$ such that $v_i, v_j \in S$. We have that $v_1, v_{n-2}, v_{n-1}, v_n \in S$. Thus $S = V$. Hence $\mathrm{con}(D) = n-2$. (ii) If $3 \leqslant k \leqslant n-3$ and $n \geqslant 6$ then let $D = (V, E)$ be the digraph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_1, v_i), (v_i, v_k): i = 2, \ldots, k-1\} \cup \{(v_k, v_1), (v_k, v_{k+1}), (v_{k+2}, v_{k+3})\} \cup \{(v_{k+1}, v_j), (v_j, v_1): j = k+2, \ldots, n\}$. Then $|E| = 2(k-2) + 3 + 2(n-k-1) = 2n-3$ and $\{v_1, v_2, \ldots, v_k\}$ is a convex set. For $1 \leqslant i \leqslant k < j \leqslant n$, $v_1, v_k, v_{k+1} \in I[v_i, v_j]$; that is, $[\{v_i, v_j\}] = V$. For $k \leqslant i < j \leqslant n$, $v_1, v_k, v_{k+1} \in I[v_i, v_j]$; that is, $[\{v_i, v_j\}] = V$. So $\{v_1, v_2, \ldots, v_k\}$ is the unique maximum convex set. Therefore $\mathrm{con}(D) = k$.

(f) For $n+1 \leqslant m \leqslant 2n-4$. Let a, b be integers with $1 \leqslant a \leqslant k-2$ and $1 \leqslant b \leqslant n-k-1$. Define $H(a, b)$ to be a strongly connected oriented graph with $V(H(a, b)) = V \cup U$ and $E(H(a, b)) = E_1 \cup E_2$ where $V = \{v_1, v_2, \ldots, v_k\}$, $U = \{u_1, u_2, \ldots, u_{n-k}\}$, $E_1 = \{(v_1, v_i), (v_i, v_{a+2}): 2 \leqslant i \leqslant a+1\} \cup \{(v_j, v_{j+1}): a+2 \leqslant j \leqslant k-1\} \cup \{(v_k, v_1)\}$ and $E_2 = \{(v_k, u_{n-k})\} \cup \{(u_j, u_{j-1}): n-k \leqslant j \leqslant b+2\} \cup \{(u_{b+1}, u_i), (u_i, v_1): 1 \leqslant i \leqslant b\}$. In $H(a, b)$, $|E(H(a, b))| = n + a + b - 1$ and V is a convex set. And $I_{H(a,b)}[v_1, v_k] = V$ and $u_1, u_2, \ldots, u_{n-k} \in I_{H(a,b)}[v_1, u_{n-k}]$. If S is a convex set containing u_i and u_j with $i < j$ then $v_1, v_k, u_{n-k} \in I_{H(a,b)}[u_i, u_j]$; that is, $S = V(H(a, b))$. If S is a convex set containing v_i and u_j then $v_1, v_k, u_{n-k} \in I_{H(a,b)}[v_i, u_j]$; that is, $S = V(H(a, b))$. Therefore V is the unique maximum convex set in H_m and $\mathrm{con}(H_m) = |V| = k$. □

For strongly connected oriented graphs with convexity number 1, we have that:

Theorem 25. *For any integers n, m with $n \geqslant 3$, $n \leqslant m \leqslant \binom{n}{2}$, there exists a strongly connected oriented graph D with n vertices, m arcs, and convexity number* 1.

Proof. First, we consider the case $m = \binom{n}{2}$. Define D_0 to be an oriented graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and arc set $\{(v_i, v_{i+1}): 1 \leqslant i \leqslant n-1\} \cup \{(v_j, v_i): 3 \leqslant i+2 \leqslant j \leqslant n\}$. Since $(v_k, v_{k+1}, v_{k+2}, v_k)$ is a directed cycle of length 3 for all $1 \leqslant k \leqslant n-2$, we have that (v_{k+1}, v_{k+2}, v_k) and (v_{k+2}, v_k, v_{k+1}) are geodesics in D_0. If a convex set of D_0 contains two consecutive vertices v_i, v_{i+1} for some $1 \leqslant i \leqslant n-1$, then it must be $V(D_0)$. Suppose that $\mathrm{con}(D_0) > 1$ and S is a convex set of D_0 with $|S| > 1$. Take $v_i, v_j \in S$ with $i < j$. Then the vertices of the geodesic $(v_i, v_{i+1}, \ldots, v_j)$ belong to S. Thus, S contains two consecutive vertices of $V(D_0)$. By the above property, $S = V(D_0)$. Hence, $\mathrm{con}(D_0) = 1$.

Second, we consider the case of $2n-2 \leqslant m \leqslant \binom{n}{2} - 1$. Let T be a subset of $\{(v_j, v_i): 1 \leqslant i \leqslant j-3 \leqslant n-3\} - \{(v_n, v_1)\}$ with $|T| = \binom{n}{2} - m$ and $D' = D_0 - T$. If S is a convex set of D' containing two distinct vertices v_i and v_j with $i < j$, then $(v_i, v_{i+1}, \ldots, v_j)$ is a geodesic in D'; that is, vertices $v_i, v_{i+1}, \ldots, v_j$ are in S. Since (v_k, v_{k+1}, v_{k-1}) and (v_{k+1}, v_{k-1}, v_k) are geodesics in D' for $2 \leqslant k \leqslant n-1$ and $v_i, v_{i+1} \in S$, $S = V(D')$. Hence D' is an oriented graph with m arcs and $\mathrm{con}(D') = 1$.

Finally, the case $n \leqslant m \leqslant 2n-3$. For $1 \leqslant k \leqslant n-2$, define $D_k = (V, E)$ with $V = \{u_0, u_1, \ldots, u_{n-1}\}$ and $E = \{(u_0, u_i), (u_i, u_{k+1}): 1 \leqslant i \leqslant k\} \cup \{(u_i, u_{i+1}): k+1 \leqslant i \leqslant n-2\} \cup \{(u_{n-1}, u_0)\}$. Then D_k is strongly connected and $|E| = n + k - 1$. And, for each $i < j$, we have that vertices u_0 and u_{n-1} are contained in $u_j - u_i$ and $u_i - u_j$ geodesics and $I[u_0, u_{n-1}] = V$; that is, $\mathrm{con}(D_k) = 1$. □

Consider the strongly connected oriented graphs with convexity number $n-1$. By Lemma 13, there is no strongly connected tournament of order $n \geqslant 3$ with convexity number $n-1$; and, the convexity number of each directed cycle is 1. So we have the following theorem.

Theorem 26. *For any integers n and m with* $n \geqslant 4$, *there exists a strongly connected oriented graph D with n vertices, m arcs, and convexity number* $n-1$ *if, and only if,* $n+1 \leqslant m \leqslant \binom{n}{2} - 1$.

Proof. By Lemma 13, there is no strongly connected tournament D of order $n \geqslant 3$ with $\mathrm{con}(D) = n-1$. So, we assume that $n+1 \leqslant m \leqslant \binom{n}{2} - 1$. If $n+1 \leqslant m \leqslant \binom{n-1}{2} + 2 = \binom{n}{2} - (n-3)$ then it is easy to construct a strongly connected oriented graph D' with $n-1$ vertices and $m-2$ arcs. Let $(u, v) \in E(D')$ and w be a new vertex. Define an oriented graph $D = (V, E)$ with $V = V(D') \cup \{w\}$ and $E = E(D') \cup \{(u, w), (w, v)\}$. Then D is a strongly connected graph with n vertices and m edges, and w being a transitive vertex of D. Thus $\mathrm{con}(D) = n-1$.

If $\binom{n}{2} - (n-4) \leqslant m \leqslant \binom{n}{2} - 1$ and $n \geqslant 5$ then, let $r = \binom{n}{2} - m$, define a strongly connected graph $D = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_a, v_b): 1 \leqslant a < b \leqslant n-2\} \cup \{(v_c, v_d): r+1 \leqslant c \leqslant n-2 \text{ and } n-1 \leqslant d \leqslant n\} \cup \{(v_{n-1}, v_i): 1 \leqslant i \leqslant r\} \cup \{(v_n, v_{n-1})\}$. We observe that D is a strongly connected oriented graph with n vertices and m edges, and v_n is a transitive vertex of D. Therefore $\mathrm{con}(D) = n-1$. □

For the remained cases, we have the following result.

Theorem 27. *Suppose D is a strongly connected oriented graph with n vertices and m edges. Then*

(1) *if* $n = 3$ *and* $m = 3$ *then* $\mathrm{con}(D) = 1$;
(2) *if* $n = 4$ *and* $m = 4$ *then* $\mathrm{con}(D) = 1$;
(3) *if* $n = 4$ *and* $m = 5$ *then* $\mathrm{con}(D) = 1$ *or* 3;
(4) *if* $n = 4$ *and* $m = 6$ *then* $\mathrm{con}(D) = 1$.

Proof. (1) If D is a strongly connected oriented graph with 3 vertices and 3 edges then D is directed cycle of length 3. Thus $\mathrm{con}(D) = 1$.

(2) If D is a strongly connected oriented graph with 4 vertices and 4 edges then D is directed cycle of length 4. Thus $\mathrm{con}(D) = 1$.

(3) If D is a strongly connected oriented graph with 4 vertices and 5 edges then the underlying graph of D is isomorphic to $K_4 - \{e\}$ for some edge $e \in E(K_4)$. Let the underlying graph D be the $G = (V, E)$ with $V = \{x_1, x_2, x_3, x_4\}$ and $E = \{x_i x_j : i < j\} - \{x_1 x_4\}$. Without loss of generality, $(x_2, x_1), (x_1, x_3)$ are in $E(D)$. If x_1 is a transitive vertex in D then (x_2, x_3) is in $E(D)$. Thus, $(x_3, x_4), (x_4, x_2)$ are in $E(D)$ and $\mathrm{con}(D) = 3$. If x_1 is not a transitive vertex in D then (x_3, x_1) is in $E(D)$. If x_4 is not a transitive vertex in D then $\mathrm{con}(D) = 1$; otherwise, for x_4 being a transitive vertex, $\mathrm{con}(D) = 3$.

(4) By Theorem 15, $S_{SC}(K_4) = \{1\}$. Then $\mathrm{con}(D) = 1$. □

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